

K₁ OF A p -ADIC GROUP RING II. THE DETERMINANTAL KERNEL SK₁.

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ABSTRACT. We describe the group $\mathrm{SK}_1(R[G])$ for group rings $R[G]$ where G is an arbitrary finite group and where the coefficient ring R is a p -adically complete Noetherian integral domain of characteristic zero which admits a lift of Frobenius and which also satisfies a number of further mild conditions. Our results extend previous work of R. Oliver who obtained such results for the valuation rings of finite extensions of the p -adic field.

1. INTRODUCTION

For an arbitrary (unital) ring S , the group $\mathrm{K}_1(S)$ is defined as

$$\mathrm{K}_1(S) = \mathrm{GL}(S)/\mathrm{E}(S)$$

where $\mathrm{GL}(S)$ denotes the (infinite) general linear group of S and $\mathrm{E}(S)$ denotes the subgroup of elementary matrices over S . In this article we continue our study of K_1 of a group ring $R[G]$ for a finite group G , where the coefficient ring R is a p -adically complete ring. The present paper is the second in a series of two papers: the first paper [CPT1] dealt with the determinantal image of $\mathrm{K}_1(R[G])$, whereas this paper is concerned with the kernel of the determinant map, $\mathrm{SK}_1(R[G])$. It is interesting to note that recently there has been considerable resurgence of interest in K_1 of group rings with higher dimensional rings of coefficients in equivariant Iwasawa theory (see for instance [FK], [K], [Ka1-3] and [RW1,2]).

The conditions we impose on R are slightly stronger than in [CPT1]: R will always denote an Noetherian integral domain of finite Krull dimension with field of fractions, denoted N , of characteristic zero, and N^c will denote a chosen algebraic closure of N . We then have a map, which we denote Det ,

$$(1.1) \quad \mathrm{Det} : \mathrm{K}_1(R[G]) \rightarrow \mathrm{K}_1(N^c[G]) = \bigoplus_{\chi} N^{c \times}$$

where the direct sum extends over the irreducible N^c -valued characters of G . We write $\mathrm{SK}_1(R[G]) = \ker(\mathrm{Det})$, so that we have the exact sequence

$$(1.2) \quad 1 \rightarrow \mathrm{SK}_1(R[G]) \rightarrow \mathrm{K}_1(R[G]) \rightarrow \mathrm{Det}(\mathrm{K}_1(R[G])) \rightarrow 1.$$

Date: May 6, 2013.

Chinburg is supported by NSF Grant # DMS11-00355.

Pappas is supported by NSF Grant # DMS11-02208.

2010 Mathematics Subject Classification. 19B28(primary), and 19C99, 16E20 (secondary).

In this paper we complete our study by presenting a number of results on $\mathrm{SK}_1(R[G])$. One of our main motivations has been the generalization of Fröhlich's theory to higher dimensional schemes over \mathbb{Z} ; in particular, some of the results of this paper are used, in a crucial manner, in the proof of the adelic Riemann-Roch theorem of [CPT2]. The results for $\mathrm{Det}(K_1(R[G]))$ were obtained in [CPT1] by a generalization of the group logarithm, as developed in [T] (see also [CR2] and [F]). Here, our results for $\mathrm{SK}_1(R[G])$ are, to a large extent, obtained by developing the ideas of R. Oliver as presented in his sequence of papers [O1-4] and his book [O5], which also contain his own independent description of the group logarithm. One of the key new achievements of this paper is the definition of a new group logarithm which extends the previous group logarithm for p -groups to all finite groups (see Sect. 6.c). This is achieved by the use of Adams operations on the group $\mathrm{Det}(K_1(R[G]))$, which are constructed by generalizing the work of P. Cassou-Noguès and the third named author in [CNT].

Let p be a prime number. Throughout this paper, unless explicitly indicated to the contrary, we shall assume that:

Standing Hypotheses.

- (i) the natural map $R \rightarrow \varprojlim_n R/p^n R$ is an isomorphism, so that R is p -adically complete;
- (ii) R supports a Frobenius lift, i.e a \mathbb{Z}_p -algebra endomorphism $F = F_R : R \rightarrow R$ with the property that for all $r \in R$

$$F(r) \equiv r^p \pmod{pR};$$

- (iii) pR is a prime ideal of R ;
- (iv) $\mathrm{SK}_1(R \otimes_{\mathbb{Z}_p} W) = \{1\}$ for the valuation ring W of any finite non-ramified extension of \mathbb{Q}_p .

Examples of such rings R are: the valuation ring of a non-ramified extension of the p -adic field \mathbb{Q}_p ; the p -adic completion of the polynomial ring $\mathbb{Z}_p[T_1, \dots, T_n]$ over \mathbb{Z}_p

$$\mathbb{Z}_p\langle\langle T_1, \dots, T_n \rangle\rangle = \varprojlim_n \mathbb{Z}_p[T_1, \dots, T_n]/(p^n);$$

the p -adic completion of the ring of formal Laurent series over \mathbb{Z}_p

$$\mathbb{Z}_p\{\{T_1, \dots, T_n\}\} = \varprojlim_n \mathbb{Z}_p((T_1, \dots, T_n))/(p^n).$$

Here $\mathbb{Z}_p((T_1, \dots, T_n)) = \mathbb{Z}_p[[T_1, \dots, T_n]][T_1^{-1}, \dots, T_n^{-1}]$. In each of the latter two examples we may take $F(T) = T^p$. To see that for these choices of R we have $\mathrm{SK}_1(R \otimes_{\mathbb{Z}_p} W) = \{1\}$, first recall that by a result of Bloch ([Bl, p. 354], $\mathrm{SK}_1(S) = \{1\}$ if S is a localization of a regular local Noetherian (commutative) domain. This shows that $\mathrm{SK}_1(W) = \{1\}$, $\mathrm{SK}_1(W((T_1, \dots, T_n))) = \{1\}$. Now an approximation argument as in the proof of Theorem 2.4 shows that $\mathrm{SK}_1(W\{\{T_1, \dots, T_n\}\}) = \{1\}$. Similarly, the result for the case when $R = W\langle\langle T_1, \dots, T_n \rangle\rangle$ comes from the argument in Lemma 2.15 of [CPT2] that uses results of L. Gruson.

Unless stated to the contrary we shall extend F to an R -algebra endomorphism of the group ring $R[G]$ by setting $F(rg) = F(r)g$ for all $r \in R$, $g \in G$.

Since R is p -adically complete, $R[G]$ is of course also p -adically complete and the Jacobson radical of R necessarily contains pR . If $\{u_n\}$ is a p -adically convergent sequence of units converging to r in $R[G]$, then r will be congruent to u_n modulo p for large n , and so r is a unit, and hence $R[G]^\times$ is also seen to be p -adically complete. (See Remark 1.1 in [CPT1].)

The group $R[G]^\times = \text{GL}_1(R[G])$ embeds into $\text{GL}(R[G])$ as diagonal matrices with all non-leading diagonal terms equal to 1. We recall Theorem 1.2 of [CPT1]:

Theorem 1.1. *Let R be as in the Standing Hypotheses. The inclusion $R[G]^\times \subset \text{GL}(R[G])$ induces an equality*

$$\text{Det}(R[G]^\times) = \text{Det}(\text{GL}(R[G])) = \text{Det}(K_1(R[G]))$$

in the following two circumstances:

- (a) *when G is a p -group;*
- (b) *when G is an arbitrary finite group, if R is in addition normal.*

For the purposes of this article we shall be particularly interested in the completed K -groups

$$\widehat{K}_i(R[G]) = \varprojlim_n K_i(R[G]/(p^n))$$

for $i = 1, 2$. From Proposition 1.5.1 in [FK] we know that, if all the quotient rings $R_n = R/p^n R$ are finite, then the natural map $K_1(R[G]) \rightarrow \widehat{K}_1(R[G])$ is an isomorphism. (See also the work of C. T. C. Wall in [W1] and [W2].) The following two results are a generalization of their result. For these we only need to assume that R is p -adically complete, i.e that $R \simeq \varprojlim_n R/p^n R$:

Theorem 1.2. *Assume that R is a p -adically complete Noetherian integral domain with fraction field of characteristic zero. Then the natural map $K_1(R[G]) \rightarrow \widehat{K}_1(R[G])$ is an isomorphism.*

Theorem 1.3. *Assume that R is a p -adically complete Noetherian integral domain with fraction field of characteristic zero. For $p^n > 2$ the reduction map $K_2(R[G]) \rightarrow K_2(R_n[G])$ is surjective.*

The above leads us to formulate:

Problem 1.4. *Assume R is as above. Let m be an arbitrary non-negative integer. Are the reduction maps $K_m(R[G]) \rightarrow K_m(R_n[G])$ surjective? Is the natural map $K_m(R[G]) \rightarrow \widehat{K}_m(R[G])$ an isomorphism?*

Remark. By the long exact sequence of K -theory (see for instance (2.1) in Sect. 2), a positive answer to the first question above is equivalent to the injectivity of natural map $K_{m-1}(R[G], p^n) \rightarrow K_{m-1}(R[G])$.

Our first result for SK_1 concerns the image of $\mathrm{SK}_1(R[G])$ in $\mathrm{K}_1(R_n[G])$, which we denote by $\mathrm{SK}_1(R[G])_n$; for the purposes of this result we again only need R to be p -adically complete:

Theorem 1.5. *Assume that R is a p -adically complete Noetherian integral domain with fraction field of characteristic zero. If $p^n > 2$, then $\mathrm{SK}_1(R[G])$ maps isomorphically onto $\mathrm{SK}_1(R[G])_n$ under the natural map $\mathrm{K}_1(R[G]) \rightarrow \mathrm{K}_1(R_n[G])$.*

The key to our study of $\mathrm{SK}_1(R[G])$ is the group logarithm (see Section 3 in [CPT1] for details). Suppose for the moment that G is a p -group. We let $I_G = I(R[G])$ denote the augmentation ideal $\ker(R[G] \rightarrow R)$; let $\mathcal{A}(R[G]) = \ker(R[G] \rightarrow R[G^{\mathrm{ab}}])$, let C_G denote the set of conjugacy classes of G and let $\phi : R[G] \rightarrow R[C_G]$ be the R -linear map obtained by mapping each group element to its conjugacy class. We write $\mathrm{K}'_1(R[G], I_G)$ for the image of $\mathrm{K}_1(R[G], I_G)$ in $\mathrm{K}_1(R[G])$. We denote the Whitehead group $\mathrm{K}'_1(R[G], I_G)/\mathrm{Im}(G)$ by $\mathrm{Wh}_G(R)$, or Wh_G when R is clear from the context. Using Theorems 3.15 and 3.17 in [CPT1] we obtain the exact sequence:

$$1 \rightarrow \frac{\mathrm{Det}(1 + I_G)}{\mathrm{Det}(G)} \rightarrow \phi(I_G) \xrightarrow{\omega} G^{\mathrm{ab}} \otimes_{\mathbb{Z}} \frac{R}{(1 - F)R} \rightarrow 1$$

where ω is the logarithmic derivative map of Proposition 3.18 loc. cit.; using Theorem 1.1 (a) we obtain the further exact sequence

$$1 \rightarrow \mathrm{SK}_1(R[G]) \rightarrow \mathrm{Wh}_G(R) \rightarrow \frac{\mathrm{Det}(1 + I_G)}{\mathrm{Det}(G)} \rightarrow 1.$$

These two exact sequences may then be spliced together to give the four term exact sequence

$$(1.3) \quad 1 \rightarrow \mathrm{SK}_1(R[G]) \rightarrow \mathrm{Wh}_G(R) \rightarrow \phi(I_G) \rightarrow G^{\mathrm{ab}} \otimes_{\mathbb{Z}} \frac{R}{(1 - F)R} \rightarrow 1.$$

Let $H_2^{\mathrm{ab}}(G, \mathbb{Z})$ denote the subgroup of the Schur multiplier $H_2(G, \mathbb{Z})$ generated by the images under corestriction of the $H_2(A, \mathbb{Z})$ for all abelian subgroups A of G , and set $\overline{H}_2(G, \mathbb{Z}) = H_2(G, \mathbb{Z})/H_2^{\mathrm{ab}}(G, \mathbb{Z})$. We may then use Oliver's construction (see Section 3 for details) to construct from this exact sequence a map

$$(1.4) \quad \Theta_{R[G]} : \mathrm{SK}_1(R[G]) \rightarrow \frac{R}{(1 - F)R} \otimes \overline{H}_2(G, \mathbb{Z}).$$

We extend Oliver's proof to show:

Theorem 1.6. *For a p -group G and for a ring R , which satisfies the Standing Hypotheses, the map $\Theta_{R[G]}$ is an isomorphism.*

In Sections 5 and 6 we extend our results from p -groups to arbitrary finite groups G . The formulation of our general result uses ideas from [IV]. We let G_r denote the set of p -regular elements in G ; we let $R[G_r]$ denote the free R -module on the elements of G_r and we let Ψ be the R -module endomorphism of $R[G_r]$ which is given by the rule

$$\Psi\left(\sum_{g \in G_r} a_g g\right) = \sum_{g \in G_r} F(a_g) g^p.$$

Let G act on $R[G_r]$ by conjugation on G_r ; then the homology group $H_2(G, R[G_r])$ is defined. We let $H_2^{\text{ab}}(G, R[G_r])$ denote the subgroup of $H_2(G, R[G_r])$ generated by the images under corestriction of the $H_2(A, R[A_r])$ for all abelian subgroups A of G , and we set $\overline{H}_2(G, R[G_r]) = H_2(G, R[G_r]) / H_2^{\text{ab}}(G, R[G_r])$. Then Ψ acts, via its action on $R[G_r]$, on the groups $H_2(G, R[G_r])$, $H_2^{\text{ab}}(G, R[G_r])$, $\overline{H}_2(G, R[G_r])$ and we will write $H_2(G, R[G_r])_\Psi$, $H_2^{\text{ab}}(G, R[G_r])_\Psi$, $\overline{H}_2(G, R[G_r])_\Psi$ for their groups of covariants. Then we have:

Theorem 1.7. *Let G be an arbitrary finite group and let R be a ring which satisfies the stated hypotheses, and which in addition is normal. Then there is a natural isomorphism:*

$$\Theta_{R[G]} : \text{SK}_1(R[G]) \xrightarrow{\sim} \overline{H}_2(G, R[G_r])_\Psi.$$

To analyze the right-hand side further we let $\{C_i\}_{i \in I}$ denote the set of G -conjugacy classes in G_r , let g_i be a chosen group element in C_i and set G_i denote the centralizer of g_i in G so that we have a disjoint union decomposition $G_r = \cup_i g_i^{G/G_i}$. We next consider the action of Ψ on the $\{C_i\}_{i \in I}$. We may view this action as an action on I , and we let J denote the set of orbits of Ψ on I . For $j \in J$ we let n_j denote the cardinality of j ; we then obtain a further disjoint union decomposition

$$G_r = \bigcup_{j \in J} \bigcup_{m=1}^{n_j} (g_{i_j}^{G/G_{i_j}})^{\Psi^m}$$

where i_j denotes a chosen element of the orbit j and the conjugacy class of g_{i_j} lies in C_{i_j} .

In Appendix B, we show that this decomposition affords the explicit description:

Corollary 1.8. *Under the assumptions of the above theorem we have*

$$\text{SK}_1(R[G]) \cong \overline{H}_2(G, R[G_r])_\Psi = \bigoplus_j [\overline{H}_2(G_{i_j}, \mathbb{Z}) \otimes \frac{R}{(F-1)R}].$$

Corollary 1.9. *Given two rings $R \subset S$ which satisfy the conditions of Theorem 1.7 and which have compatible lifts of Frobenius (so that the restriction of F_S to R coincides with F_R), then the natural map $\text{SK}_1(R[G])$ to $\text{SK}_1(S[G])$ is surjective, resp. an isomorphism, if the natural map*

$$\frac{R}{(1-F_R)R} \rightarrow \frac{S}{(1-F_S)S}$$

is surjective, resp. an isomorphism. If this map is injective with torsion free cokernel, then the map $\text{SK}_1(R[G])$ to $\text{SK}_1(S[G])$ is also injective.

Let W denote the ring of integers of a finite non-ramified extension of \mathbb{Q}_p and let F denote the Frobenius automorphism of W . An explicit formula for $\text{SK}_1(W[G])$ was given by R. Oliver in his series of papers [O1-4]; see also [O5]; thus Theorem 1.7 is a generalization of Oliver's result.

We now extend F to the power series ring $W[[t]]$ in an indeterminate t by setting $F(t) = t^p$. Since $(1 - F)(tW[[t]]) = tW[[t]]$ we see using the above corollary that for any finite group G the inclusion $W \subset W[[t]]$ induces an isomorphism

$$(1.5) \quad \mathrm{SK}_1(W[G]) \cong \mathrm{SK}_1(W[[t]][G]).$$

(see Proposition 5.3 in [Wi]). We also consider

$$W\langle\langle t^{-1} \rangle\rangle = \varprojlim_n (W[t^{-1}]/p^n W[t^{-1}]), \quad W\{\{t\}\} = \varprojlim_n (W((t))/p^n W((t)))$$

with similarly extended Frobenius. In Appendix B, we also show that Corollary 1.9 implies:

Corollary 1.10. *With the above notation the inclusion $W\langle\langle t^{-1} \rangle\rangle \subset W\{\{t\}\}$ induces an isomorphism $\mathrm{SK}_1(W\langle\langle t^{-1} \rangle\rangle[G]) \cong \mathrm{SK}_1(W\{\{t\}\}[G])$.*

Corollary 1.11. *With the above notation the inclusion $W[[t]] \rightarrow W\{\{t\}\}$ induces an injection $\mathrm{SK}_1(W[[t]][G]) \hookrightarrow \mathrm{SK}_1(W\{\{t\}\}[G])$.*

These results play an important role in the proof of the adelic Riemann-Roch theorem of [CPT2].

Acknowledgement: The authors would like to thank Otmar Venjakob for useful discussions.

2. GENERAL RESULTS FOR K_1

Let S denote an arbitrary unitary ring. Throughout this section I will denote a 2-sided ideal of S , which is contained in the Jacobson radical of S ; let $\overline{S} = S/I$. Recall that if $v \in S^\times$, $j \in I$, $s, t \in S$ then $v + sjt \in S^\times$. Let $M_n(S)$ denote the ring of $n \times n$ matrices with entries in S and let $M_n(I)$ denote the 2-sided ideal of $M_n(S)$ of matrices with entries in I . We set $M(I) = \varinjlim_n M_n(I)$, and we write $1 + M(I) = \varinjlim_n (1 + M_n(I))$. Note that for a matrix $x \in 1 + M_n(I)$ since all its diagonal entries are units, by left and right multiplication by elementary matrices, we can bring x into diagonal form with unit entries; thus x is invertible and so

$$1 + M(I) = \mathrm{GL}(S, I) \stackrel{\mathrm{defn}}{=} \ker(\mathrm{GL}(S) \rightarrow \mathrm{GL}(\overline{S})).$$

Note that $\mathrm{GL}(S)$ maps onto $\mathrm{GL}(\overline{S})$, since given $\overline{x} \in \mathrm{GL}_n(\overline{S})$ with inverse \overline{y} , then for lifts $x, y \in M_n(S)$ we have $xy \in 1 + M_n(I)$ and so x is invertible.

Lemma 2.1. *Suppose we are given two rings A, B and a surjection $q : A \rightarrow B$, with the property that $I = \mathrm{Ker}(q : A \rightarrow B)$ is contained in the Jacobson radical of A . Then the map $q_* : K_1(A) \rightarrow K_1(B)$ is surjective and $\ker q_*$ is generated by the image of $\mathrm{GL}(A, I)$.*

Proof. The first statement follows from the above fact that the induced map $\mathrm{GL}(A) \rightarrow \mathrm{GL}(B)$ is surjective; the second part is standard - see for instance Theorem 2.5.3 page 93 in [R]. \square

We recall the long exact sequence of K-theory (see for instance page 54 of [M])

$$(2.1) \quad K_2(S) \rightarrow K_2(\overline{S}) \rightarrow K_1(S, I) \rightarrow K_1(S) \xrightarrow{q} K_1(\overline{S}) \rightarrow 1$$

with q surjective since $\mathrm{GL}(S)$ maps onto $\mathrm{GL}(\overline{S})$. We let $E(S, I)$ denote the smallest normal subgroup of the group of elementary matrices $E(S)$ containing the elementary matrices $e_{ij}(a)$ with $a \in I$. Then we know (see for instance page 93 in [R])

$$K_1(S, I) = \frac{\mathrm{GL}(S, I)}{E(S, I)}.$$

We shall write $K'_1(S, I)$ for the image of $K_1(S, I)$ in $K_1(S)$. Note that $K_2(S) \rightarrow K_2(\overline{S})$ is surjective, if and only if, the map $K_1(S, I) \rightarrow K_1(S)$ is injective; and this is equivalent to the equality

$$(2.2) \quad \mathrm{GL}(S, I) \cap E(S) = E(S, I).$$

Recall also from [R] loc. cit. that $E(S, I)$ is a normal subgroup of $\mathrm{GL}(S)$ and

$$(2.3) \quad [\mathrm{GL}(S), E(S, I)] = [E(S), E(S, I)] = E(S, I).$$

We shall write κ_S for the map from S^\times to $K_1(S)$ given by mapping $u \in S^\times$ to the diagonal matrix which is u in the first position and 1 in all other diagonal entries and then composing with the map $\mathrm{GL}(S) \rightarrow K_1(S)$.

From Lemma 2.2 in [CPT1] we quote:

Lemma 2.2. (a) $[\mathrm{GL}(S), 1 + M(I)] = E(S, I)$.

(b) Given $g \in \mathrm{GL}(S, I)$, there exist $e_1, e_2 \in E(S, I)$, and $x \in 1 + I$ such that $g = e_1 \delta(x) e_2$ where $\delta(x)$ is the diagonal matrix with leading term x and with all non-leading diagonal terms equal to 1; hence in particular

$$\mathrm{GL}(S, I) \subset \langle E(S, I), (1 + I) \rangle$$

where we view $1 + I \subset S^\times \subset \mathrm{GL}(S, I)$ as previously. Thus, if we write κ_I for the natural map from $\mathrm{GL}(S, I)$ to $K_1(S, I)$, then $\kappa_I(1 + I) = K_1(S, I)$.

From 45.12 on page 142 of [CR2] we know:

Proposition 2.3. If S is a commutative semi-local ring, then $SK_1(S) = \{1\}$.

2.a. Proofs of Theorems 1.2, 1.3 and 1.5. In this subsection we assume only that R is p -adically complete. We let $\mathrm{SL}(R[G])$ denote the subgroup of elements in $\mathrm{GL}(R[G])$ which are trivial under Det ; more generally for a two-sided ideal I of $R[G]$ we define

$$\mathrm{SL}(R[G], I) = \mathrm{SL}(R[G]) \cap \mathrm{GL}(R[G], I).$$

We fix an ordering of the elements of the group; $G = \{g_1, \dots, g_q\}$. We start this subsection by showing:

Theorem 2.4. $E(R[G])$ is p -adically closed in $\mathrm{GL}(R[G])$ and for $p^k > 2$

$$(2.4) \quad \mathrm{SL}(R[G], p^k) = E(R[G], p^k).$$

Proof. If $m \geq 1$, we let $\overline{E_m(R[G])}$ denote the p -adic closure of $E_m(R[G])$ in $GL_m(R[G])$. We begin by showing that in order to prove $E_m(R[G]) = \overline{E_m(R[G])}$ it will suffice to show that for some integer k with $p^k > 2$ we have $SL_m(R[G], p^k) = E_m(R[G], p^k)$. To this end we first observe that we will have $E_m(R[G]) = \overline{E_m(R[G])}$ if we can show

$$E_m(R[G], p^k) = GL_m(R[G], p^k) \cap \overline{E_m(R[G])};$$

this is because if we are given $x \in \overline{E_m(R[G])}$, then, as $E_m(R[G])$ is dense in $\overline{E_m(R[G])}$, we can write $x = eg$ with $e \in E_m(R[G])$ and $g \in GL_m(R[G], p^k) \cap \overline{E_m(R[G])}$. Since

$$SL_m(R[G], p^k) \supset GL_m(R[G], p^k) \cap \overline{E_m(R[G])} \supset E_m(R[G], p^k),$$

by the above we now see that in order to show $E_m(R[G]) = \overline{E_m(R[G])}$ it will suffice to show that $SL_m(R[G], p^k) = E_m(R[G], p^k)$.

It is enough to show $SL_m(R[G], p^k) \subset E_m(R[G], p^k)$; i.e. given $x \in SL_m(R[G], p^k)$ we want to show we can write x as a finite product of elements in $E_m(R[G], p^k)$. We first note that as in [CPT1] Lemma 2.2.b given such an x we can find $e, e' \in E_m(R[G], p^k)$ so that we can write $x = ede'$ with $d \in (1 + p^k R[G]) \cap \ker(\text{Det})$. The proofs of Proposition 2.4 and Lemma 2.5 in [CPT1] apply under the hypothesis $p^k > 2$. By the last line in the proof of Lemma 2.5 in [CPT1] we have:

Lemma 2.5. *Suppose that $p^n > 2$. Given $x \in (1 + p^n R[G]) \cap \ker(\text{Det})$ we can find $\lambda_{i,n} \in R[G]$ so that*

$$x \equiv \prod_{i=1}^q [g_i, 1 + p^n \lambda_{i,n}] \pmod{p^{n+1} R[G]}.$$

(Here such products, which in the sequel are not generally commutative, are to be taken in the specified order.)

Remark. The hypotheses on n and p imply that for any $\lambda \in R[G]$ we have the congruence

$$\log(1 + p^n \lambda) \equiv p^n \lambda \pmod{p^{n+1}}$$

which is used throughout the proof of Lemma 2.5 loc. cit.. Note that we also need the commutator congruence for $h \in G, \lambda, \lambda' \in R[G]$

$$[h, 1 + p^n \lambda][h, 1 + p^n \lambda'] \equiv [h, 1 + p^n(\lambda + \lambda')] \pmod{p^{n+1}}.$$

Lemma 2.6. *For $x, y, z \in R[G]^\times$ we have the standard commutator relation*

$$[z, yx] = [z, y][z, x]^{y^{-1}}.$$

Proof. We note that

$$[z, y][z, x]^{y^{-1}} = zyz^{-1}y^{-1}yzxz^{-1}x^{-1}y^{-1} = zyxz^{-1}x^{-1}y^{-1} = [z, yx].$$

□

Lemma 2.7. *For $\lambda, \mu \in R[G]$ and $p^n > 2, m \geq 1$ we have the congruence*

$$(1 + p^n \mu)^{-1}(1 + p^m \lambda)(1 + p^n \mu) \equiv (1 + p^m \lambda) \pmod{p^{n+m} R[G]}$$

and so

$$(1 + p^m \lambda)(1 + p^n \mu)(1 + p^m \lambda)^{-1} \equiv (1 + p^n \mu) \pmod{p^{n+m} R[G]}.$$

Proof. This follows at once from the congruence

$$(1 + p^m \lambda)(1 + p^n \mu) \equiv 1 + p^n \mu + p^m \lambda \equiv (1 + p^n \mu)(1 + p^m \lambda) \pmod{p^{n+m} R[G]}$$

and the fact that $1 + p^n \mu$ and $1 + p^m \lambda$ are units. \square

Proof of Theorem 2.4. We have supposed that $p^k > 2$. Suppose that x belongs to $(1 + p^k R[G]) \cap \ker(\text{Det})$. We shall show that we can write $x = \prod_{i=1}^q [g_i, 1 + p^k \mu_i]$ with $\mu_i \in R[G]$. By Lemma 2.2.a in [CPT1] we know that each of these commutators is an element of $E(R[G], p^k)$, and so we shall be done.

For $p^n \geq p^k$ we inductively suppose that we have written

$$x \equiv \prod_{i=1}^q [g_i, 1 + p^k \mu_{i,n}] \pmod{p^{n+1} R[G]} \text{ with } \mu_{i,n} \in R[G].$$

(Note that we can use Lemma 2.5 to start the induction.) By Lemma 2.5 above we can write $x \cdot [g_q, 1 + p^k \mu_{q,n}]^{-1} \cdots [g_1, 1 + p^k \mu_{1,n}]^{-1}$ as

$$\prod_{i=1}^q [g_i, 1 + p^{n+1} \lambda_{i,n}] \pmod{p^{n+2} R[G]}$$

and so now we have written

$$x \equiv \prod_{i=1}^q [g_i, 1 + p^k \mu_{i,n}] \prod_{j=1}^q [g_j, 1 + p^{n+1} \lambda_{j,n}] \pmod{p^{n+2} R[G]}.$$

We then apply Lemma 2.7 to note that for all i, j , with $1 \leq i, j \leq q$ we have

$$[g_i, 1 + p^k \mu_{i,n}] [g_j, 1 + p^{n+1} \lambda_{j,n}] \equiv [g_j, 1 + p^{n+1} \lambda_{j,n}] [g_i, 1 + p^k \mu_{i,n}] \pmod{p^{n+2} R[G]}$$

and by Lemma 2.6 we see that for all i with $1 \leq i \leq q$ we have

$$\begin{aligned} [g_i, (1 + p^k \mu_{i,n})(1 + p^{n+1} \lambda_{j,n})] &= [g_i, (1 + p^k \mu_{i,n})] [g_i, (1 + p^{n+1} \lambda_{j,n})]^{(1 + p^k \mu_{i,n})^{-1}} \\ &\equiv [g_i, 1 + p^k \mu_{i,n}] [g_i, 1 + p^{n+1} \lambda_{i,n}] \pmod{p^{n+2} R[G]}. \end{aligned}$$

We then define

$$1 + p^k \mu_{i,n+1} = (1 + p^k \mu_{i,n})(1 + p^{n+1} \lambda_{j,n})$$

so that

$$x \equiv \prod_{i=1}^q [g_i, 1 + p^k \mu_{i,n+1}] \pmod{p^{n+2} R[G]}$$

which completes the inductive step. To conclude we note that for each i the sequence $\{\mu_{i,n}\}_n$ is a convergent sequence. \square

Proof of Theorem 1.3. By the discussion prior to equation (2.2) we know that the reduction map $K_2(R[G]) \rightarrow K_2(R_n[G])$ is surjective if

$$\text{SL}(R[G], p^n) \cap E(R[G]) = E(R[G], p^n)$$

and this follows at once from Theorem 2.4 above. \square

Proof of Theorem 1.5. We must show that for $p^n > 2$ the map $\theta_n : \text{SK}_1(R[G]) \rightarrow \text{SK}_1(R[G])_n$ is injective, since by definition θ_n is surjective. If $x \in \ker \theta_n$ then by Lemma 2.2 (b) we may write $x = \kappa(x')$ with $x' \in (1 + p^n(R[G])) \cap \ker(\text{Det}) \subset \text{SL}_m(R[G], p^n) = E_m(R[G], p^n)$. \square

Proof of Theorem 1.2. By the Bass Stable Range Theorem (see for instance 41.25 in [CR2]) we can find a sufficiently large integer m with the property that the map

$\mathrm{GL}_m(R[G])$ to $K_1(R[G])$ is surjective. Choose a positive integer n with $p^n > 2$. Recall that by the remark prior to (2.2) and by Theorem 2.4, we know that the relative K-group $K_1(R[G], p^n)$ maps injectively into $K_1(R[G])$. We have an exact sequence

$$1 \rightarrow E_m(R[G], p^n) \rightarrow \mathrm{GL}_m(R[G], p^n) \rightarrow K_1(R[G], p^n) \rightarrow 1$$

which yields (see the exact sequence (2.1))

$$1 \rightarrow \frac{E_m(R[G])}{E_m(R[G], p^n)} \rightarrow \frac{\mathrm{GL}_m(R[G])}{\mathrm{GL}_m(R[G], p^n)} \rightarrow K_1(R[G], p^n) \rightarrow 1$$

and by taking inverse limits, noting that for $n \geq j$ the morphisms

$$\frac{E_m(R[G])}{E_m(R[G], p^n)} \rightarrow \frac{E_m(R[G])}{E_m(R[G], p^j)}$$

are surjective, so that the Mittag-Leffler condition is satisfied, we get the exact sequence

$$1 \rightarrow \varprojlim_n \frac{E_m(R[G])}{E_m(R[G], p^n)} \rightarrow \mathrm{GL}_m(R[G]) \rightarrow \widehat{K}_1(R[G]) \rightarrow 1.$$

But by Theorem 2.4 we know that $\varprojlim_n E_m(R[G])/E_m(R[G], p^n)$ is just the p -adic closure $\overline{E_m(R[G])}$ and that this, in turn, coincides with $E_m(R[G])$. Hence we have shown

$$K_1(R[G]) = \frac{\mathrm{GL}_m(R[G])}{E_m(R[G])} = \widehat{K}_1(R[G]).$$

□

3. p -GROUPS

Throughout this section G denotes a finite p -group and the ring R satisfies the Standing Hypotheses.

3.a. The group logarithm. In this subsection we very briefly recall the group logarithm which is a fundamental tool for studying the determinants of units of p -adic group rings.

Recall that R is a Noetherian integral domain of finite Krull dimension with field of fractions N of characteristic zero, that F denotes a lift of Frobenius on R , and that $I(R[G])$ denotes the augmentation ideal of the group ring $R[G]$ and when R is clear from the context we shall write I_G in place of $I(R[G])$. Because $I(\mathbb{F}_p[G])$ is the Jacobson radical of the Artinian ring $\mathbb{F}_p[G]$, it follows that we can find a positive integer m such that $I(\mathbb{F}_p[G])^m = 0$. Since $I(R[G]) = R \cdot I(\mathbb{Z}_p[G])$, it therefore follows that

$$(3.1) \quad I(R[G])^m \subset pR[G].$$

Define the F -semi-linear map $\Psi : R[G] \rightarrow R[G]$ by the rule that $\Psi(rg) = F(r)g^p$. As in the Introduction $\phi : R[G] \rightarrow R[C_G]$ denotes the R -linear map given by sending each group element to its conjugacy class. Write $\overline{\Psi} : R[C_G] \rightarrow R[C_G]$ for the F -semi-linear map induced by Ψ .

We define the group logarithm $\mathcal{L} : 1 + I(R[G]) \rightarrow N[C_G]$ by the rule that for $x \in I(R[G])$

$$(3.2) \quad \begin{aligned} \mathcal{L}(1-x) &= (p - \bar{\Psi}) \circ \phi(\log(1-x)) = \phi((p - \Psi)(\log(1-x))) \\ &= -\phi\left(\sum_{n \geq 1} \frac{px^n}{n} - \sum_{n \geq 1} \frac{\Psi(x^n)}{n}\right) \end{aligned}$$

which is seen to converge to an element of $N[C_G]$ by (3.1) above. Note that the map $\mathcal{L} = \mathcal{L}_F$ depends on the chosen lift of Frobenius F ; we shall therefore have to be particularly careful when using such group logarithms for different coefficient rings which could have incompatible lifts of Frobenius.

From Lemma 3.2 in [CPT1] we recall:

Lemma 3.1. *For a character χ of G and for $x \in I(R[G])$*

$$\chi(\phi(\log(1+x))) = \log(\text{Det}(1+x)(\chi))$$

and

$$\chi(\mathcal{L}(1+x)) = \log[\text{Det}(1+x)(p\chi) \cdot \text{Det}(1+F(x))(-\psi^p\chi)]$$

where ψ^p denotes the p -th Adams operation on virtual characters of G , which is defined by the rule $\psi^p\chi(g) = \chi(g^p)$ for $g \in G$. \square

As per Corollary 3.3 loc. cit. from this we may deduce:

Corollary 3.2. *If $\text{Det}(1+x) = 1$, then $\phi(\log(1+x)) = 0$, and so $\mathcal{L}(1+x) = 1$. Thus there is a unique map $\nu : \text{Det}(1 + I(R[G])) \rightarrow N[C_G]$ such that $\mathcal{L} = \nu \circ \text{Det}$.* \square

The main result for the group logarithm is (see Theorem 3.4.a in [CPT1]):

Theorem 3.3. *We have $\mathcal{L}(1 + I_G) \subset p\phi(I_G) \subset pR[C_G]$.*

We now recall a number of results on the images under \mathcal{L} of various subgroups of $1 + I(R[G])$. For our first such result we set $\mathcal{A}(R[G]) = \text{Ker}(R[G] \rightarrow R[G^{\text{ab}}])$ and note that obviously $\mathcal{A}(R[G]) \subset I(R[G])$. Then from Theorem 3.5 loc. cit. we have:

Theorem 3.4. *We have $\mathcal{L}(1 + \mathcal{A}(R[G])) = p\phi(\mathcal{A}(R[G]))$.*

We now assume G to be non-abelian; then, since G is a p -group, by considering the lower central series of G , we may choose a commutator $c = [\gamma, \delta]$ which has order p and which lies in the centre of G ; it therefore follows that

$$(1-c)R[G] \subset \mathcal{A}(R[G]).$$

From Lemma 3.8 and Lemma 3.10 loc. cit. we recall:

Lemma 3.5. *For $n \geq 1$, if c is a commutator as above, we have $\mathcal{L}(1 + (1-c)^n R[G]) = p\phi((1-c)^n R[G])$.* \square

If we now assume that c is again a central element order of p but that now it cannot be written as a commutator in G , then from Lemmas 3.12 and 3.13 in [CPT1] we have the following two lemmas:

Lemma 3.6. *If c cannot be written as a commutator in G , then multiplication by c permutes the elements of C_G without fixed points and the kernel of multiplication by $1 - c$ on $R[C_G]$ is generated by elements of the form $\sum_{i=0}^{p-1} c^i \phi(g)$ for $g \in G$. \square*

Lemma 3.7. *Suppose again that c cannot be written as a commutator in G ; then*

$$\mathcal{L}(1 + (1 - c)I(R[G])) \supseteq p\phi((1 - c)I(R[G])) + p^2\phi(1 - c)R$$

and

$$\frac{p\phi((1 - c)R[G])}{\mathcal{L}(1 + (1 - c)R[G])} \cong \frac{R}{(1 - F)R + pR}$$

induced by mapping $p\phi((1 - c)rg) \mapsto r \bmod pR$ for $r \in R, g \in G$. \square

The relative K -group $K_1(R[G], I_G)$ was defined in Sect. 2. Recall $K'_1(R[G], I_G)$ denotes the image of $K_1(R[G], I_G)$ in $K_1(R[G])$. We then define the Whitehead group $\text{Wh}(R[G])$ to be $K'_1(R[G], I_G)/\text{Im}(G)$ and the determinantal Whitehead group $\text{Wh}'(R[G])$ is defined to be $\text{Det}(1 + I_G)/\text{Det}(G)$. From Theorems 3.14 and 3.17 in [CPT1] we have the exact sequence

$$(3.3) \quad 1 \rightarrow \text{Det}(G) \rightarrow \text{Det}(1 + I_G) \xrightarrow{v} \phi(I_G) \xrightarrow{\omega} \frac{R}{(1 - F)R} \otimes G^{\text{ab}} \rightarrow 1.$$

where we write $v = p^{-1}\nu$. Recall that by hypothesis $\text{SK}_1(R) = \{1\}$, and so by using the commutative diagram with exact horizontal sequence

$$\begin{array}{ccccccccc} 1 & \rightarrow & \text{SK}_1(R) & \rightarrow & K_1(R) & \rightarrow & \text{Det}(R^\times) & \rightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \rightarrow & \text{SK}_1(R[G]) & \rightarrow & K_1(R[G]) & \rightarrow & \text{Det}(R[G]^\times) & \rightarrow & 1 \end{array}$$

we see that $\text{SK}_1(R[G]) \subseteq K'_1(R[G], I_G)$. Noting that the image of G in $K'_1(R[G], I_G) \subset K_1(R[G])$ is isomorphic to $\text{Det}(G)$, using the snake lemma we get the two exact sequences

$$(3.4) \quad 1 \rightarrow \text{SK}_1(R[G]) \rightarrow \text{Wh}(R[G]) \rightarrow \text{Wh}'(R[G]) \rightarrow 1$$

$$(3.5) \quad 1 \rightarrow \text{SK}_1(R[G]) \rightarrow \text{Wh}(R[G]) \xrightarrow{\Gamma} \phi(I_G) \rightarrow \frac{R}{(1 - F)R} \otimes G^{\text{ab}} \rightarrow 1.$$

where Γ is obtained by composing Det with $p^{-1}\nu$. Suppose now that we have an exact sequence of p -groups

$$1 \rightarrow H \xrightarrow{\iota} \tilde{G} \xrightarrow{\alpha} G \rightarrow 1.$$

Our study of SK_1 will be based on the detailed study of the two groups

$$(3.6) \quad \mathcal{H} = \ker(\alpha_*) = \ker(\text{SK}_1(R[\tilde{G}]) \rightarrow \text{SK}_1(R[G])),$$

$$(3.7) \quad \mathcal{C} = \text{coker}(\alpha_*) = \text{coker}(\text{SK}_1(R[\tilde{G}]) \rightarrow \text{SK}_1(R[G])).$$

3.b. SK₁ for *p*-groups.

Lemma 3.8. *If G is an abelian p -group, then $\text{SK}_1(R[G]) = \{1\}$.*

Proof. As previously, we know that $\text{SK}_1(R[G]) \subset K'_1(R[G], I_G)$ and so we may assume that $x \in \text{SL}(R[G])$ has the same image in $K_1(R[G])$ as $x' \in \text{GL}(R[G], I_G)$. Let $x \in \text{SL}(R[G])$ be represented by $x' \in \text{GL}(R[G], I_G)$. By Lemma 2.2 (b) we may write $x' = e_1 \delta e_2$ with $e_i \in E(R[G], I_G)$, $\delta \in 1 + I_G$. Since G is abelian, Det is an isomorphism on $R[G]^\times$; since $\text{Det}(\delta) = 1$, we conclude that $\delta = 1$, and so $x' \in E(R[G], I_G)$. \square

We may therefore henceforth assume G to be non-abelian. As previously we choose a central element c of G which has order p and which is a commutator; that is to say we can write

$$c = [h, g] = hgh^{-1}g^{-1}, \quad c^p = 1$$

and we set $\overline{G} = G/\langle c \rangle$. We define

$$\mathcal{B} = \mathcal{B}(R[G]) = \ker(R[G] \rightarrow R[\overline{G}]) = (1 - c)R[G].$$

From Lemma 4.2 in [CPT1] we recall the exact sequence

$$1 \rightarrow \text{Det}(1 + (1 - c)R[G]) \rightarrow \text{Det}(1 + I(R[G])) \rightarrow \text{Det}(1 + I(R[\overline{G}])) \rightarrow 1$$

and so and so using Corollary 3.2, (3.3), Lemma 3.5 and noting that $\text{Det}(G) \cap \text{Det}(1 + (1 - c)R[G]) = \{1\}$ (by projecting into G^{ab}), we get the further exact sequence

$$(3.8) \quad 0 \rightarrow p\phi(\mathcal{B}) \rightarrow \text{Det}(1 + I(R[G])) \rightarrow \text{Det}(1 + I(R[\overline{G}])) \rightarrow 1.$$

Using the exact sequence

$$1 \rightarrow \text{Det}(1 + I(R[G])) \rightarrow \text{Det}(R[G]^\times) \rightarrow R^\times \rightarrow 1$$

first for G , and then with G replaced by \overline{G} , and using the snake lemma we get

$$0 \rightarrow p\phi(\mathcal{B}) \rightarrow \text{Det}(R[G]^\times) \rightarrow \text{Det}(R[\overline{G}]^\times) \rightarrow 1.$$

Proposition 3.9. (a) *The quotient map $q_* : \text{SK}_1(R[G]) \rightarrow \text{SK}_1(R[\overline{G}])$ is surjective;*
 (b) *let $\mathcal{H} = \mathcal{H}(R[G]) = \ker(\text{SK}_1(R[G]) \rightarrow \text{SK}_1(R[\overline{G}]))$, then*

$$\mathcal{H} = \{ \kappa(x) \mid x \in \text{GL}(R[G], \mathcal{B}) \text{ with } \text{Det}(x) = 1 \}.$$

Proof. This follows at once from the commutative diagram

$$\begin{array}{ccccccc} 1 \rightarrow & \mathcal{H} & \hookrightarrow & \text{SK}_1(R[G]) & \xrightarrow{q_*} & \text{SK}_1(R[\overline{G}]) & \rightarrow 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \rightarrow & K'_1(R[G], \mathcal{B}) & \hookrightarrow & K_1(R[G]) & \rightarrow & K_1(R[\overline{G}]) & \rightarrow 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \rightarrow & \text{Det}(K_1(R[G], \mathcal{B})) & \hookrightarrow & \text{Det}(K_1(R[G])) & \rightarrow & \text{Det}(K_1(R[\overline{G}])) & \rightarrow 1. \end{array}$$

Here the middle row is exact: indeed, the right-hand arrow is surjective, as \mathcal{B} is contained in the Jacobson radical of $R[G]$; and the fact that $K'_1(R[G], \mathcal{B})$ is equal to the kernel of this map follows from (2.2). By Lemma 2.2 (b) we know that $\text{Det}(K_1(R[G], \mathcal{B})) = \text{Det}(1 + \mathcal{B})$ and by (3.8) we can identify $\text{Det}(1 + \mathcal{B})$ with $p\phi(\mathcal{B})$; the exactness of the middle row then follows from the exact sequence given prior to

the statement of the proposition. Therefore, by the snake lemma, we know that the top row is exact, and in particular q_* is surjective. \square

Next we apply the reduction map $K_1(R[G]) \rightarrow K_1(R_n[G])$ to the exact sequence

$$1 \rightarrow \mathcal{H} \rightarrow SK_1(R[G]) \rightarrow SK_1(R[\overline{G}]) \rightarrow 1.$$

Recall from the Introduction that $SK_1(R[G])_n$ denotes the image of $SK_1(R[G])$ in $K_1(R_n[G])$. Writing \mathcal{H}_n for the image of \mathcal{H} in $K_1(R_n[G])$, we shall now show:

Proposition 3.10. *The following sequence is exact:*

$$1 \rightarrow \mathcal{H}_n \rightarrow SK_1(R[G])_n \rightarrow SK_1(R[\overline{G}])_n \rightarrow 1.$$

Before proving this proposition we first require a preliminary result:

Lemma 3.11. *For brevity we shall write $((1 - c))$ and (p^n) for the two-sided $R[G]$ -ideals $(1 - c)R[G]$ and $p^n R[G]$.*

- (a) $\text{Det}(1 + ((1 - c))) \cong p\phi(((1 - c)))$.
- (b) For $n \geq 1$, $\text{Det}(1 + (p^n(1 - c))) \cong p^{n+1}\phi((1 - c))$.
- (c) For $n \geq 2$, $\nu(\text{Det}(1 + (p^n))) \subset \phi((p^n))$.
- (d) For $n \geq 2$

$$\nu \circ \text{Det}(1 + ((1 - c))) \cap \nu \circ \text{Det}(1 + (p^n)) = \nu \circ \text{Det}(1 + (p^n(1 - c)))$$

and

$$\text{Det}(1 + ((1 - c))) \cap \text{Det}(1 + (p^n)) \subset \text{Det}(1 + (p^{n-1}(1 - c))).$$

Proof. First note again that, because $\psi^p((1 - c)) = 1 - c^p = 0$, it follows that $p\phi \circ \log$ coincides with the group logarithm on $1 + ((1 - c))$. By Corollary 3.2 we know that $p \cdot \phi \circ \log = \nu \circ \text{Det}$ on $1 + ((1 - c))$; and so by Lemma 3.5

$$\text{Det}(1 + ((1 - c))) \cong p\phi((1 - c)),$$

and for $n \geq 1$ the result (b) now follows from Proposition 2.4 in [CPT1]

$$\text{Det}(1 + (p^n(1 - c))) \cong p^{n+1}\phi((1 - c)).$$

To see (c) we note that for $n \geq 2$

$$\mathcal{L}(1 - p^n x) = -\phi\left(\sum_{m \geq 1} \frac{p^{1+mn} x^m}{m} - \sum_{m \geq 1} \frac{\Psi(p^{nm} x^m)}{m}\right) \subset p^n R[C_G].$$

The result (d) then follows since we know that

$$p\phi((p^{n-1})) \cap p\phi((1 - c)) = p\phi((p^{n-1}(1 - c))).$$

To conclude, if $x \in \text{Det}(1 + ((1 - c))) \cap \text{Det}(1 + (p^n))$ we have $\nu(x) = \nu(y)$ for some $y \in \text{Det}(1 + p^{n-1}(1 - c)r)$, for some $r \in R[G]$; the result then follows since ν is injective on $\text{Det}(1 + ((1 - c)))$. \square

Proof of Proposition 3.10. Consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \rightarrow & \mathcal{H} & \rightarrow & \mathrm{SK}_1(R[G]) & \xrightarrow{q_*} & \mathrm{SK}_1(R[\overline{G}]) & \rightarrow & 1 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 1 & \rightarrow & \mathcal{H}_n & \xrightarrow{i} & \mathrm{SK}_1(R[G])_n & \xrightarrow{j} & \mathrm{SK}_1(R[\overline{G}])_n & \rightarrow & 1 \end{array}$$

in which, by Proposition 3.9, we know the top row to be exact. By definition i is an inclusion map and hence is injective; j is surjective since γ and q_* are surjective; clearly $j \circ i = 0$. Therefore it remains to show $\ker(j) \subset \mathrm{Im}(i)$. For brevity given $x \in \mathrm{GL}(R[G])$, we shall write \overline{x} for its image in $\mathrm{GL}(R[\overline{G}])$. Now consider $x \in \mathrm{SK}_1(R[G])$ with $j \circ \beta(x) = 1$; then $\gamma(\overline{x}) = \gamma \circ q_*(x) = 1$ and so we can find $a_i, b_i \in \mathrm{GL}(R[G])$ such that

$$\overline{x} = \prod_i [\overline{a}_i, \overline{b}_i] \cdot (1 + p^n \overline{\lambda}_1)$$

with $\lambda_1 \in \mathrm{M}(R[G])$; furthermore by Lemma 2.2 (b) we can find elementary matrices $d_1, d_2 \in \mathrm{E}(R[G], p^n)$ and $\lambda_2 \in R[G]^\times$ so that $1 + p^n \lambda_1 = d_1(1 + p^n \lambda_2)d_2$, where we view $R[G]^\times$ as a subgroup of $\mathrm{GL}(R[G])$ in the usual way. Thus we can deduce that for some $\mu \in \mathrm{M}(R[G])$

$$x = \prod_i [a_i, b_i] \cdot d_1(1 + p^n \lambda_2)d_2 \cdot (1 + (1 - c)\mu).$$

Using Lemma 2.2 (b) we can write

$$1 + (1 - c)\mu = e_1(1 + (1 - c)\mu')e_2$$

with $e_i \in \mathrm{E}(R[G], I)$ and $\mu' \in R[G]$. Therefore, taking determinants and using the fact that $\mathrm{Det}(x) = 1$, we get

$$\mathrm{Det}(1 + (1 - c)\mu) = \mathrm{Det}(1 + (1 - c)\mu') \in \mathrm{Det}(1 + (p^n)) \cap \mathrm{Det}(1 + ((1 - c)))$$

and Lemma 3.11 shows that we can write

$$(3.9) \quad \mathrm{Det}(1 + (1 - c)\mu) = \mathrm{Det}(1 + p^n(1 - c)\xi)$$

for some $\xi \in R[G]$. Therefore

$$(3.10) \quad 1 + (1 - c)\mu = (1 + p^n(1 - c)\xi)\tau$$

for some element $\tau \in \mathrm{GL}(R[G], I)$ with the property that $\mathrm{Det}(\tau) = 1$; hence $\tau \in \mathcal{H}$, and so we are done, since $\beta(x)$ is equal to the class of $i(\alpha(\tau)) \in \mathrm{Im}(i)$, as required. \square

3.b.1. Some commutator identities. We now need to analyze $\mathcal{H}_R = \mathcal{H}(R[G])$ in greater detail. The various groups \mathcal{H}_R will provide the inductive building blocks for $\mathrm{SK}_1(R[G])$ and, by Lemma 2.2 (b), the elements of $\mathrm{SK}_1(R[G])$ are represented by elements of $R[G]^\times$ which have trivial determinant but which are not themselves products of commutators (in $\mathrm{GL}(R[G])$). This then leads us to establish a number of commutator relations and congruences, which we then use in subsequent subsections. All these appear either explicitly or implicitly in Oliver's work (see for example [O1]). One important point of this paper is that most of these identities from Oliver's work (which regards rings of integers in finite extensions of \mathbb{Q}_p) continue to hold when the

ring R satisfies our standing hypotheses. In what follows, we explain this and also reorganize some of the material from Oliver's papers.

As in the previous parts of this section we continue to suppose that G is a p -group. First we suppose G to be non-abelian and, as previously, c denotes a central commutator of order p ,

$$c = [h, g] = hgh^{-1}g^{-1}$$

and we put $\overline{G} = G/\langle c \rangle$; hence

$$(3.11) \quad cg = hgh^{-1}$$

also $c = g^{-1}cg = g^{-1}hgh^{-1}$ and so $ch = g^{-1}hg$. We then note that for $n > 0$, and $\lambda \in R$ we have

$$(3.12) \quad \begin{aligned} [g^{-1}h, 1 - \lambda(g-h)^n] &= g^{-1}h(1 - \lambda(g-h)^n)h^{-1}g(1 - \lambda(g-h)^n)^{-1} \\ &= (1 - \lambda c^n(g-h)^n)(1 - \lambda(g-h)^n)^{-1}. \end{aligned}$$

Noting the entirely obvious identity in non-commuting indeterminates X and Y

$$(3.13) \quad (1 - YX)(1 - X)^{-1} = 1 + (1 - Y)X(1 - X)^{-1},$$

setting $X = \lambda(g-h)^n$, $Y = c^n$, and observing that $(1 - c^n) = (1 + c + \cdots + c^{n-1})(1 - c)$, we get

$$(3.14) \quad [g^{-1}h, 1 - \lambda(g-h)^n] = 1 + \lambda(1 - c^n)(g-h)^n(1 - \lambda(g-h)^n)^{-1}$$

so this is now equal to

$$1 + n(1 - c)[\lambda(g-h)^n + \lambda^2(g-h)^{2n} + \lambda^3(g-h)^{3n} + \cdots] \bmod (1 - c)^2.$$

We now require two further such commutator congruences, both of whose proofs are entirely similar, but which are in fact slightly easier.

The first is for $n > 0$. For such n we have

$$[h, 1 - \lambda(1 - c)^{n-1}g] = (1 - \lambda(1 - c)^{n-1}hgh^{-1})(1 - \lambda(1 - c)^{n-1}g)^{-1}$$

and, setting $X = \lambda(1 - c)^{n-1}g$ and $Y = hgh^{-1}g^{-1}$ in (3.13), we get

$$(3.15) \quad \begin{aligned} [h, 1 - \lambda(1 - c)^{n-1}g] &= 1 + \lambda(1 - c)^{n-1}(g - hgh^{-1})(1 - \lambda(1 - c)^{n-1}g)^{-1} \\ &= 1 + \lambda(1 - c)^n g(1 - \lambda(1 - c)^{n-1}g)^{-1} \end{aligned}$$

and if $n > 1$

$$(3.16) \quad \equiv 1 + \lambda(1 - c)^n g \bmod (1 - c)^{n+1}.$$

For the second relation we again take $n > 0$ and we no longer insist that g and h be chosen such that $c = [h, g]$. We then have

$$[h, 1 - \lambda(1 - c)^n g] = (1 - \lambda(1 - c)^n hgh^{-1})(1 - \lambda(1 - c)^n g)^{-1}$$

and by the identity (3.13) with $X = \lambda(1 - c)^n g$, $Y = hgh^{-1}g^{-1}$, we have

$$[h, 1 - \lambda(1 - c)^n g] = 1 + \lambda(1 - c)^n (g - hgh^{-1})(1 - \lambda(1 - c)^n g)^{-1}$$

$$(3.17) \quad \equiv 1 + \lambda(1 - c)^n(g - hgh^{-1}) \pmod{(1 - c)^{n+1}}.$$

3.b.2. *Analysis of the group $\mathcal{H}(R[G])$.* As always we suppose that R satisfies the Standing Hypotheses; note that it is here that we shall use the hypothesis that pR is a prime ideal of R .

We start our analysis of $\mathcal{H}_R = \mathcal{H}(R[G])$ by using the above commutator identities taken together with the logarithmic methods of 3.a.

Recall that \mathcal{H}_R denotes $\ker(\mathrm{SK}_1 R[G] \rightarrow \mathrm{SK}_1(R[\overline{G}]))$. We shall now analyze \mathcal{H}_R by means of the following groups c.f. page 203 in [O1]:

$$\begin{aligned} \mathcal{I}_R &= \ker[(1 + (1 - c)R[G]) \xrightarrow{\alpha} K_1(R[G])] \\ \mathcal{J}_{n,R} &= \ker[(1 + (1 - c)^n R[G]) \xrightarrow{\alpha_n} \mathrm{Det}(R[G]^\times)] \end{aligned}$$

where the map α is induced by the map $\mathrm{GL}(R[G]) \rightarrow K_1(R[G])$ and α_n is induced by the determinant. We shall frequently write \mathcal{J}_R for $\mathcal{J}_{1,R}$. Obviously we have the inclusion $\mathcal{I}_R \subset \mathcal{J}_R$. When the ring R is clear from the context we shall just write \mathcal{I}, \mathcal{J} in place of $\mathcal{I}_R, \mathcal{J}_R$. Recall that N denotes the field of fractions of R .

Lemma 3.12. *Define*

$$\mathcal{S}_G = \{g \in G \mid c^m g \text{ are all conjugate to } g \text{ for all } 0 \leq m < p\}$$

and let D_G denote the set of conjugacy classes $\phi(\mathcal{S}_G)$. Then, for $n \geq 1$, the kernel of $\phi : (1 - c)^n N[G] \rightarrow N[C_G]$ consists of the N -linear span of the following two kinds of element:

- (**Type 1**) $(1 - c)^n g$ for $g \in \mathcal{S}_G$;
- (**Type 2**) $(1 - c)^n(g - hgh^{-1})$ for $g, h \in G$.

Proof. Suppose that $n \geq 1$, as in the statement of the lemma. The central subgroup $\langle c \rangle$ of order p acts naturally by multiplication on the conjugacy classes C_G . Thus $N[C_G]$ is a permutation $N\langle c \rangle$ -module, and as such it is isomorphic to a direct sum of copies of $N\langle c \rangle$ (on which $(1 - c)^n$ acts faithfully, since N has characteristic zero) and a sum of copies of N on which $(1 - c)^n$ acts as 0. The latter summands form exactly the N -linear span $N[D_G]$ of D_G . We conclude that for every $\mu \in N[G]$ there is an element $\lambda \in N[\mathcal{S}_G]$ such that the image $\phi(\mu - \lambda)$ of $\mu - \lambda$ in $N[C_G]$ is annihilated by $(1 - c)^n$ if and only if $\phi(\mu - \lambda) = 0$. We have

$$(1 - c)^n \mu = (1 - c)^n(\mu - \lambda) + (1 - c)^n \lambda$$

where $\phi((1 - c)^n \lambda) = 0$. Thus $\phi((1 - c)^n \mu) = 0$ if and only if $\phi((1 - c)^n(\mu - \lambda)) = (1 - c)^n \phi((\mu - \lambda)) = 0$, and this forces $\phi(\mu - \lambda) = 0$. Thus $\mu - \lambda$ is an N -linear combination of elements of the form $g - hgh^{-1}$ with $g, h \in G$, and this implies the lemma. \square

Remark. In the sequel we shall write

$$T_{R,1} = \sum_{g \in \mathcal{S}_G} Rg, \quad T_{R,2} = \sum_{g,h} R(g - hgh^{-1}).$$

We refer to the former as elements of Type 1 and the latter as elements of Type 2.

Proposition 3.13. *For $n \geq 1$*

$$\mathcal{J}_n = \{u \in 1 + (1 - c)^n R[G] \mid \phi \circ \log(u) = 0 \text{ in } N[C_G]\}.$$

Hence, by Lemma 3.12, \mathcal{J}_n is the set of those $u \in 1 + (1 - c)^n R[G]$ with the property that $\log u$ is of the form $(1 - c)^n \xi$ with ξ a sum of R -linear multiples of elements of Types 1 and 2.

Proof. Using Lemma 3.5 and reasoning as in the first line of the proof of Lemma 3.11, we have a diagram

$$\begin{array}{ccc} 1 + (1 - c)R[G] & \xrightarrow{\text{Det}} & \text{Det}(1 + (1 - c)R[G]) \\ & \searrow & \downarrow v \\ & & \phi((1 - c)R[G]) \end{array}$$

where the diagonal map is $\phi \circ \log$ and where the induced map $v = \frac{1}{p}\nu$ is an isomorphism. Thus $\ker(\phi \circ \log) = \ker(p\phi \circ \log) = \ker(\text{Det})$, as required. \square

Lemma 3.14. *We have $\mathcal{H} \cong \mathcal{J}/\mathcal{I}$.*

Proof. (See page 205 in [O1]). We note again that by Lemma 2.2 (b) we know that $\text{SK}_1(R[G])$ is contained in the image of $\kappa_{R[G]}$. For brevity we again write $((1 - c))$ for the ideal $(1 - c)R[G]$ and we consider the commutative diagram: with exact rows:

$$\begin{array}{ccccccccc} 1 & \rightarrow & \mathcal{I} & \rightarrow & 1 + ((1 - c)) & \xrightarrow{\alpha} & \kappa_{I_G}(1 + I(R[G])) & \rightarrow & \kappa_{I_{\overline{G}}}(1 + I(R[\overline{G}])) & \rightarrow & 1 \\ & & \downarrow & & \downarrow = & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & \mathcal{J} & \rightarrow & (1 + ((1 - c))) & \xrightarrow{\alpha_1} & \text{Det}(1 + I(R[G])) & \rightarrow & \text{Det}(1 + I(R[\overline{G}])) & \rightarrow & 1. \end{array}$$

The exactness of the top row follows from the exactness of the middle row in the diagram of the proof of Proposition 3.9 using Lemma 2.2 (b) to identify

$$\kappa_{R[G]}(R[G]^\times) = K_1(R[G]), \quad \kappa_{R[\overline{G}]}(R[\overline{G}]^\times) = K_1(R[\overline{G}]).$$

The second row is exact by (3.8). We then split the long exact sequences up into short exact sequences and apply the snake lemma to the left-hand short exact sequences to get

$$\mathcal{J}/\mathcal{I} \cong \ker(\text{Im}(\alpha) \rightarrow \text{Im}(\alpha_1));$$

while by the right-hand short exact sequences we get that

$$\ker(\text{Im}(\alpha) \rightarrow \text{Im}(\alpha_1)) \cong \mathcal{H}.$$

\square

We now analyze the groups \mathcal{I} and \mathcal{J} by means of filtrations. We will show:

Lemma 3.15. *For $n \geq 2$*

$$(3.18) \quad \mathcal{I} \cap (1 + ((1 - c)^n)) = \mathcal{J} \cap (1 + ((1 - c)^n)) \bmod ((1 - c)^{n+1}).$$

Proof. (Note that this proof is very similar to that of Theorem 1.5 with the role of p^n replaced by $(1-c)^n$.)

We show (3.18) for all $n \geq 2$. The inclusion \subset results from the fact that $\mathcal{I} \subset \mathcal{J}$. Conversely, to show the inclusion \supset , we consider a typical generator u of $\mathcal{J} \cap (1 + ((1-c)^n))$. Note that since $n \geq 2$ we know

$$\log(u) \equiv u - 1 \pmod{(1-c)^{n+1}}.$$

We wish to show that $u \in \mathcal{I} \pmod{((1-c)^{n+1})}$. Since $\text{Det}(u) = 1$, by Proposition 3.13 we know that we can write

$$\log(u) = \sum_i n_i (1-c)^n g_i + \sum_j m_j (1-c)^n (h_j - h_j^{l_j})$$

where $n_i, m_j \in R$, the former latter right hand terms are of Type 1 and $l_j \in G$ so that the latter right hand terms are all of Type 2. Thus for some $k_i \in G$ we have

$$[k_i, g_i] = c.$$

We therefore have shown that

$$u \equiv 1 + \sum_i n_i (1-c)^n g_i + \sum_j m_j (1-c)^n (h_j - h_j^{l_j}) \pmod{(1-c)^{n+1}}$$

and so we have shown that we have the congruence $\pmod{(1-c)^{n+1}}$

$$u \equiv \prod_i (1 + \sum_i n_i (1-c)^n g_i) \prod_j (1 + m_j (1-c)^n (h_j - h_j^{l_j})).$$

Now by (3.16), since

$$[k_i, g_i^{k_i}] = [k_i, k_i^{-1} g_i k_i] = [k_i, g_i c^{-1}] = [k_i, g_i] = c,$$

we know that

$$[k_i, 1 - n_i (1-c)^{n-1} g_i] \equiv 1 + n_i (1-c)^n g_i \pmod{(1-c)^{n+1}}$$

$$\begin{aligned} [1 + m_j (1-c)^n h_j, l_j^{-1}] &= (1 + m_j (1-c)^n h_j) l_j^{-1} (1 + m_j (1-c)^n h_j)^{-1} l_j \\ (3.19) \quad &\equiv (1 + m_j (1-c)^n h_j) (1 - m_j (1-c)^n h_j^{l_j}) \pmod{(1-c)^{n+1}} \\ &\equiv (1 + m_j (1-c)^n (h_j - h_j^{l_j})) \pmod{(1-c)^{n+1}} \end{aligned}$$

and so

$$u \equiv \prod_i [k_i, 1 - n_i (1-c)^{n-1} g_i] \prod_j [1 + m_j (1-c)^n h_j, l_j^{-1}] \pmod{(1-c)^{n+1}}.$$

This then shows that $u \in \mathcal{I} \pmod{((1-c)^{n+1})}$ as required. \square

In summary, if we let $\overline{\mathcal{I}}$, resp. $\overline{\mathcal{J}}$, denote the image of \mathcal{I} , resp. \mathcal{J} , in

$$(1 + ((1-c))) \pmod{((1-c)^2)},$$

then, by Lemma 3.15, the above shows that

$$(3.20) \quad \mathcal{H} \cong \overline{\mathcal{J}} / \overline{\mathcal{I}}.$$

Next we analyze the groups $\overline{\mathcal{I}}$ and $\overline{\mathcal{J}}$ via logarithmic methods in order to evaluate the right-hand term in (3.20). We begin by analyzing $\overline{\mathcal{J}}$ and then follow this with an analysis of $\overline{\mathcal{I}}$.

From Lemma 3.6 in [CPT1] we quote the elementary congruence:

Lemma 3.16. *We have $(1 - c)^p \equiv -p(1 - c) \pmod{p(1 - c)^2}$.*

We write $\overline{R} = R/pR$. Note that for any $r \in R[G]$, if $(1 - c)r = 0$, then r is a multiple of $\sum_{n=0}^{p-1} c^n$. Using this and the above lemma we have the two commutative diagrams with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & (1 - c)^{p-1}\overline{R}[G] & \rightarrow & \overline{R}[G] & \xrightarrow{1-c} & \overline{R}[G]/((1 - c)) & \rightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \rightarrow & (1 - c)^{p-1}\overline{R}[G] & \rightarrow & (1 - c)\overline{R}[G] & \rightarrow & (1 - c)\overline{R}[G]/(1 - c)^2 & \rightarrow & 0, \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \rightarrow & (1 - c)^2\overline{R}[G] & \rightarrow & (1 - c)\overline{R}[G] & \rightarrow & (1 - c)\overline{R}[G]/(1 - c)^2\overline{R}[G] & \rightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \rightarrow & (1 - c)^2R[G] & \rightarrow & (1 - c)R[G] & \rightarrow & (1 - c)R[G]/(1 - c)^2R[G] & \rightarrow & 0 \\ & & \uparrow & & \uparrow & & & & \\ & & (1 - c)^pR[G] & = & (1 - c)pR[G] & & & & \end{array}$$

which afford isomorphisms

$$\frac{1 + (1 - c)R[G]}{1 + (1 - c)^2R[G]} \cong \frac{(1 - c)R[G]}{(1 - c)^2R[G]} \cong \overline{R}[G] \pmod{(1 - c)},$$

and we refer abusively to the latter isomorphism as “ $(1 - c)^{-1}$ ”.

We then define L to be the composite map

$$L : \frac{1 + (1 - c)R[G]}{1 + (1 - c)^2R[G]} \cong \frac{(1 - c)R[G]}{(1 - c)^2R[G]} \cong \overline{R}[G]/((1 - c)) \xrightarrow{\phi} \overline{R}[C_G]/(1 - c)\overline{R}[C_G].$$

We next consider the scaled logarithm map

$$\lambda : \frac{(1 + ((1 - c)))}{(1 + ((1 - c)^2))} \rightarrow \overline{R}[C_G]/(1 - c)\overline{R}[C_G].$$

given by the rule:

$$\lambda = \phi \circ (1 - c)^{-1} \circ \log = (1 - c)^{-1} \phi \circ \log$$

where $\lambda(1 + (1 - c)x) = \phi \circ (1 - c)^{-1} \circ \log(1 + (1 - c)x) \pmod{((1 - c), p)}$.

We now obtain

$$(3.21) \quad \lambda(1 + (1 - c)u) \equiv \phi(u) - \phi(u^p) \equiv \phi(u) - \Psi \circ \phi(u) \pmod{((1 - c), p)}$$

so that, using the map L defined above, we can view λ as the composite

$$(3.22) \quad \lambda : \frac{1 + (1 - c)R[G]}{1 + (1 - c)^2R[G]} \xrightarrow{L} \overline{R}[C_G] \pmod{(1 - c)} \xrightarrow{1 - \Psi} \overline{R}[C_G] \pmod{(1 - c)}.$$

Proposition 3.17. *For brevity we write T_1 and T_2 for $T_{R,1}$ and $T_{R,2}$ which were defined after Lemma 3.12, and let \tilde{T}_i denote $T_{R,i} \bmod p$. Assume that c is a central element of order p which is a commutator of G ; then*

- (a) $\mathcal{H} \cong \overline{\mathcal{J}}/\overline{\mathcal{I}} = \lambda(\overline{\mathcal{J}})/\lambda(\overline{\mathcal{I}})$;
- (b) $\lambda(\overline{\mathcal{J}}) = \phi(\text{Im}(1 - \Psi)\overline{R}[G] \cap (\tilde{T}_1 + \tilde{T}_2))$;
- (c) $\lambda(\overline{\mathcal{J}}) = \text{Im}(1 - \Psi)\overline{R}[C_G] \cap \overline{R}[D_G]$.

Proof. We start by observing that as per the reasoning in Lemma 3.12

$$\ker(L) = 1 + (1 - c)T_{R,2} \bmod (1 - c)^2.$$

Indeed let $1 + (1 - c)r$ denote an element of $\ker L$, so that

$$\phi(r) = p\phi(a) + (1 - c)\phi(b)$$

for some $a, b \in R[G]$. By Lemma 3.16 for some $u \in R\langle c \rangle^\times$ we have

$$(1 - c)(r - pa - (1 - c)b) = (1 - c)r - (1 - c)^p ua - (1 - c)^2 b$$

and so if we let $r' = r + (1 - c)^{p-1} ua + (1 - c)b$, then $(1 - c)r' = (1 - c)(r - pa - (1 - c)b)$ and so we see that $(1 - c)r$ and $(1 - c)r'$ represent the same element in the domain of L , and moreover

$$\phi((1 - c)r') = \phi((1 - c)(r - pa - (1 - c)b)) = 0.$$

Therefore by Lemma 3.12 we know that $(1 - c)r'$ may be written as an R -linear combination of elements of S_G and terms of the form $g - hgh^{-1}$ for $g, h \in G$; and so we are reduced to the case where r' is a linear sum of elements in S_G and it then follows that

$$\phi(r') \bmod p \in \overline{R}[D_G] \cap (1 - c)\overline{R}[C_G] = (0)$$

and so r' is indeed a sum of terms in $T_{R,2}$.

Next we note that since $1 - \Psi$ is the identity on $(1 - c)\overline{R}[C_G]$ we have the equalities

$$\begin{aligned} \ker(\overline{R}[C_G] \bmod (1 - c) \xrightarrow{1 - \Psi} \overline{R}[C_G] \bmod (1 - c)) &= \ker(1 - \Psi : \overline{R}[C_G] \rightarrow \overline{R}[C_G]) \\ &= \ker(1 - \Psi : \overline{R} \rightarrow \overline{R}) = \mathbb{F}_p \end{aligned}$$

with the penultimate equality holding because Ψ is nilpotent on the augmentation ideal $I(\overline{R}[G])$ and with the final equality holding because, by hypothesis, \overline{R} is an integral domain of characteristic p . First note that since c is a commutator, $\text{Det}(c) = 1$ and so $c \in \mathcal{J}$. The above shows that $\ker(\lambda)/\ker(L)$ identifies with the \mathbb{F}_p -line $\langle c \rangle \bmod (1 - c)^2$ and so we have now shown that

$$\ker(\lambda) = 1 + (1 - c)T_{\overline{R},2} + (1 - c)\mathbb{F}_p \bmod (1 - c)^2.$$

Moreover, by (3.19) and (3.22) we know that $1 + (1 - c)T_{\overline{R},2} \subset \overline{\mathcal{I}}$. And also, since c is a commutator, we know $1 + (1 - c)\mathbb{F}_p \bmod (1 - c)^2 = \langle c \rangle \bmod (1 - c)^2 \subset \overline{\mathcal{I}}$; hence $\ker \lambda \subset \overline{\mathcal{I}}$, and therefore we have an isomorphism

$$\overline{\mathcal{J}}/\overline{\mathcal{I}} = \lambda(\overline{\mathcal{J}})/\lambda(\overline{\mathcal{I}})$$

which together with (3.20) proves (a).

Recall also that in Lemma 3.12 D_G was defined as the subset of conjugacy classes in C_G which are fixed under multiplication by c . Consider $1 + u(1 - c) \bmod (1 - c)^2 \in \overline{\mathcal{J}}$. By Proposition 3.13 we know that

$$\log(1 + u(1 - c)) \equiv (1 - c)(\xi_1 + \xi_2) \bmod (1 - c)^2$$

for $\xi_i \in T_{R,i}$; while by (3.22) we know

$$\log(1 + u(1 - c)) \equiv (1 - c)(u - u^p) \bmod (1 - c)^2;$$

hence we have shown:

$$\lambda(\overline{\mathcal{J}}) \subset \phi \left((1 - \Psi) \overline{R}[G] \cap (\tilde{T}_1 + \tilde{T}_2) \right).$$

To complete the proof of (b) it remains to prove the inclusion \supseteq . Suppose now we are given $u \in R[G]$, $n \in T_1 + T_2$ with

$$\tilde{u} - \Psi(\tilde{u}) = \tilde{n} \in \text{Im}(1 - \Psi) \overline{R}[G] \cap (\tilde{T}_1 + \tilde{T}_2).$$

It follows from (3.22) that

$$(3.23) \quad \log(1 + (1 - c)u) = (1 - c)n + (1 - c)^2 e$$

for some $e \in R[G]$. By Lemma 3.5, using the argument in Lemma 3.11, we see that \mathcal{L} and $p\phi \circ \log$ coincide on $1 + (1 - c)R[G]$ (see for instance Lemma 3.8 of [CPT1]), and we know that we can find $f \in R[G]$ such that

$$(3.24) \quad \phi \circ \log(1 + (1 - c)^2 f) = \phi((1 - c)^2 e).$$

If we set $x = (1 + (1 - c)u)(1 + (1 - c)^2 f)^{-1}$; then, firstly, we note

$$\phi \circ \log(x) = \phi((1 - c)n)$$

and so $\lambda(x) = \phi(n) \bmod (1 - c)$. Secondly, as $n \in T_1 + T_2$, it follows that $\phi((1 - c)n) = 0$, and so by (3.22) we deduce that

$$\phi \circ \log(x) = \phi((1 - c)n) = 0.$$

But we have seen previously in the proof of Lemma 3.11 that

$$\ker(\phi \circ \log) = \ker(\text{Det}) \text{ on } 1 + (1 - c)R[G].$$

It therefore follows that $\text{Det}(x) = 1$, and so $x \in \mathcal{J}$, as required.

Finally, in order to prove part (c), we must show that the right-hand expressions in (b) and (c) coincide. The inclusion of the right-hand side of (b) in the right-hand side of (c) is obvious. Conversely, given $x \in (1 - \Psi) \overline{R}[C_G] \cap \overline{R}[D_G]$ we may write it as

$$x = (1 - \Psi) \sum_{\gamma \in C_G} m_\gamma \gamma = \sum_{\delta \in D_G} n_\delta \delta$$

with $m_\gamma, n_\delta \in \overline{R}$. We then choose group elements $\gamma', \delta' \in G$ with the property that $\phi(\gamma') = \gamma$, $\phi(\delta') = \delta$, and observe that for some $l_{g,h} \in \overline{R}$

$$y \stackrel{\text{defn}}{=} (1 - \Psi) \sum m_\gamma \gamma' = \sum n_\delta \delta' + \sum_{g,h} l_{g,h} (g - hgh^{-1});$$

hence $y \in (1 - \Psi)\overline{R}[G] \cap (\tilde{T}_1 + \tilde{T}_2)$; and, so by construction, we have $\phi(y) = x$. \square

Corollary 3.18. *If c is not a commutator, then D_G is empty and we see that $\lambda(\overline{\mathcal{T}}) = 0$, and so in this case the map $\text{SK}_1(R[G]) \rightarrow \text{SK}_1(R[\overline{G}])$ is injective. (C.f. Proposition 7 in [O1].)*

Proof. If $c \notin [G, G]$ then D_G is empty and so we are done by Prop. 3.17 (c) above. Hence, by Prop. 3.17, \mathcal{H} is trivial and so, by the very definition of \mathcal{H} , the map $\text{SK}_1(R[G]) \rightarrow \text{SK}_1(R[\overline{G}])$ is injective. \square

We now have the following straightforward generalization of Lemma 8 in [O1]:

Lemma 3.19. *If c is a commutator, so that we can write $c = [g, h]$ for $g, h \in G$, then for any $r \in \overline{R} = R/pR$:*

- (i) $r(g - g^k) \in \lambda(\overline{\mathcal{I}})$ for any integer k prime to p ;
- (ii) $(r - r^p)g \in \lambda(\overline{\mathcal{I}})$;
- (iii) for g, h as given with $h \in \mathcal{S}_G$, $r(g - h) \in \lambda(\overline{\mathcal{I}})$.

Proof. First we note that by restricting to the subgroup generated by g, h we may assume that \overline{G} is abelian; note that in this case the set $\mathcal{S} = \mathcal{S}_G$ contains no p -th powers.

Let Λ denote the \mathbb{F}_p -vector space

$$(3.25) \quad \Lambda = \lambda(\overline{\mathcal{I}}) + \sum_{k \notin \mathcal{S}} \overline{R}\phi(k)$$

and note that of course this is an internal direct sum, as $\lambda(\overline{\mathcal{I}}) \subset \lambda(\overline{\mathcal{T}}) \subset \overline{R}[D_G]$.

In order to prove the lemma we claim that it suffices to show that the 3 types of term, stated in the lemma, lie in Λ and all have trivial projection into the right-hand factor of the direct sum (3.25). We demonstrate this is indeed the case by showing that each of the three terms lies in $\tilde{R}[D_G]$. By hypothesis $c = [g, h]$, and so $c = ghg^{-1}h^{-1}$ and hence $c^{-1} = hgh^{-1}g^{-1}$. Therefore

$$c^{-1}g = {}^h g \text{ and } c^{-k}g^k = {}^h g^k;$$

therefore, for k coprime to p , we know that $g^k \in \mathcal{S}$. This then shows that the terms listed in (i) and (ii) lie in $\overline{R}[D_G]$, and then same is true for the terms in (iii) since we are given $h, g \in \mathcal{S}$.

To conclude we must now show that each of the three types of element listed in the lemma belongs to Λ . For an element $g \in G$ we write \overline{g} for the image of g in \overline{G} . (Note that this is different to the notational convention used by Oliver in [O1].) Thus, as above, we know that the subgroup of G generated by g^p, h^p is abelian and central.

Proof of (iii): From identity (3.14) we know that for any $n > 0$, $\mu \in R$

$$(3.26) \quad [g^{-1}h, 1 - \mu(g-h)^n] \equiv 1 + n(1-c)\{\mu(g-h)^n + \mu^2(g-h)^{2n} \dots\} \pmod{(1-c)^2}.$$

As we have seen above, we may drop p -th powers and so consider only the n coprime to p and thereby get that

$$\{\mu(g-h)^n + \mu^2(g-h)^{2n} + \dots\} \in \Lambda.$$

Note first that $\phi(g-h)^p = \phi(g^p) - \phi(h^p)$ in $\overline{R}[C_G]$ and so lies in $\sum_{k \notin S} \overline{R}\phi(k)$; next, since $(g-h)^N = 0$ in $\overline{R}[G]$ for $N \gg 0$, we may argue by downwards induction to get that $\mu(g-h)^n \in \Lambda$ for all $n > 0$.

Proof of (i): As above we know that for k coprime to p we can find an integer m so that we have the equality

$$(3.27) \quad [h^m, g^{p+k}] = [h^m, g^k] = c.$$

Using part (iii) above, we know that for any $k > 0$ we have

$$(3.28) \quad \mu g^k(1-g^p) = \mu(g^k - h^m) - \mu(g^{p+k} - h^m) \in \Lambda$$

and this implies that for any $r \geq p$ we have

$$(3.29) \quad \mu g(1-g)^r = \mu g(1-g)^{r-p}(1-g)^p = \mu g(1-g)^{r-p}(1-g^p) \in \Lambda.$$

So for any $r > 1$

$$\begin{aligned} [h^{-1}, 1 + \mu(1-g)^r] &= (1 + \mu h^{-1}(1-g)^r h)(1 + \mu(1-g)^r)^{-1} \\ &= (1 + \mu(1-cg)^r)(1 + \mu(1-g)^r)^{-1} \\ &= (1 + \mu(1-g)^r - \mu(1-g)^r + \mu(1-cg)^r)(1 + \mu(1-g)^r)^{-1} \\ &= 1 + (\mu(1-cg)^r - \mu(1-g)^r)(1 + \mu(1-g)^r)^{-1}. \end{aligned}$$

Writing

$$(1-cg)^r = ((1-g) + g(1-c))^r = \sum_{i=0}^r \binom{r}{i} (1-g)^{r-i} g^i (1-c)^i$$

we get

$$[h^{-1}, 1 + \mu(1-g)^r] \equiv 1 + \mu r(1-c)(1-g)^{r-1} g [1 - \mu(1-g)^r]^{-1} \pmod{(1-c)^2}.$$

Omitting p -th powers, as previously, we suppose that r is coprime to p and get that

$$(3.30) \quad \mu g(1-g)^{r-1} - \mu^2 g(1-g)^{2r-1} + \mu^3 g(1-g)^{3r-1} + \dots \in \Lambda.$$

Now by (3.28) above we already know that for $ir-1 \geq p$ we have $\mu^i g(1-g)^{ir-1} \in \Lambda$, and so, again by downwards induction, we get

$$(3.31) \quad \mu r g(1-g)^{r-1} = \mu r g(1-g)^{r-2}(1-g) \in \Lambda$$

for all $2 \leq r \leq p-1$ and hence

$$\mu g \equiv \mu g^2 \equiv \dots \equiv \mu g^{p-1} \pmod{\Lambda}.$$

The result then follows by (3.28) and so $\mu(g-h^k) \in \Lambda$ for any k coprime to p .

Proof of (ii): When we take $r = 1$ in (3.30) above we get

$$\mu g - \mu^2 g(1 - g) + \mu^3 g(1 - g)^2 + \cdots + \mu^p g(1 - g)^{p-1} \in \Lambda.$$

By part (i) we get

$$\begin{aligned} \mu g + \mu^p g(1 - g)^{p-1} &\equiv \mu g + \mu^p (g + g^2 + \cdots + g^{p-1}) \\ &= \mu g - \mu^p g + \mu^p ((g^2 - g) + \cdots + (g^{p-1} - g)) \in \Lambda. \end{aligned}$$

By part (i) we know $\mu^p (g - g^k) \in \Lambda$ for each $1 \leq k \leq p - 1$ and so $(\mu - \mu^p)g \in \Lambda$, as required. \square

The following is a straightforward generalization of Theorem 1 in [O1]:

Theorem 3.20. *With the above notation $\mathcal{H}(R[G]) = \ker(\mathrm{SK}_1(R[G]) \rightarrow \mathrm{SK}_1(R[\overline{G}]))$ is generated by the elements $\exp(r(1 - c)(g - g'))$ for elements $g, g' \in \mathcal{S}_G$.*

The following follows easily from the above theorem by arguing by induction on the order of the group G (see Proposition 9 in [O1] for details):

Proposition 3.21. *If G contains a normal abelian subgroup A with the property that G/A is cyclic then $\mathrm{SK}_1(R[G]) = \{1\}$.*

3.c. **Oliver's map $\Theta_{R[G]}$.** Recall that $\mathrm{Wh}(R[G])$ and $\mathrm{Wh}'(R[G])$ were defined in 3.a and were shown to sit in exact sequences (see (3.4) and (3.5)):

$$(3.32) \quad 1 \rightarrow \mathrm{SK}_1(R[G]) \rightarrow \mathrm{Wh}(R[G]) \rightarrow \mathrm{Wh}'(R[G]) \rightarrow 1$$

$$(3.33) \quad 1 \rightarrow \mathrm{SK}_1(R[G]) \rightarrow \mathrm{Wh}(R[G]) \xrightarrow{\Gamma_G} \phi(I_G) \rightarrow \frac{R}{(1 - F)R} \otimes G^{\mathrm{ab}} \rightarrow 1.$$

Suppose now that we have an exact sequence of finite p -groups

$$1 \rightarrow H \xrightarrow{\iota} \tilde{G} \xrightarrow{\alpha} G \rightarrow 1.$$

Our study of SK_1 will be based on the detailed study of the two groups

$$(3.34) \quad \mathcal{K} = \ker(\alpha_*) = \ker(\mathrm{SK}_1(R[\tilde{G}]) \rightarrow \mathrm{SK}_1(R[G])),$$

$$(3.35) \quad \mathcal{C} = \mathrm{coker}(\alpha_*) = \mathrm{coker}(\mathrm{SK}_1(R[\tilde{G}]) \rightarrow \mathrm{SK}_1(R[G])).$$

The above exact sequence (3.33) affords a commutative diagram with exact rows:

$$(3.36) \quad \begin{array}{ccccccc} 1 & \rightarrow & \mathrm{SK}_1(R[H]) & \rightarrow & \mathrm{Wh}(R[H]) & \xrightarrow{\Gamma_H} & \phi(I_H) & \xrightarrow{\omega_H} & R_F \otimes H^{\mathrm{ab}} & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow \phi(\iota) & & \downarrow 1 \otimes \iota^{\mathrm{ab}} & & \\ 1 & \rightarrow & \mathrm{SK}_1(R[\tilde{G}]) & \rightarrow & \mathrm{Wh}(R[\tilde{G}]) & \xrightarrow{\Gamma_{\tilde{G}}} & \phi(I_{\tilde{G}}) & \xrightarrow{\omega_{\tilde{G}}} & R_F \otimes \tilde{G}^{\mathrm{ab}} & \rightarrow & 1 \\ & & \downarrow \alpha_* & & \downarrow \mathrm{Wh}(\alpha) & & \downarrow & & \downarrow 1 \otimes \pi^{\mathrm{ab}} & & \\ 1 & \rightarrow & \mathrm{SK}_1(R[G]) & \rightarrow & \mathrm{Wh}(R[G]) & \xrightarrow{\Gamma_G} & \phi(I_G) & \xrightarrow{\omega_{\tilde{G}}} & R_F \otimes G^{\mathrm{ab}} & \rightarrow & 1 \end{array}$$

where we write $R_F = R/(1 - F)R$ for the covariants of the action of F on R .

Reasoning as in the snake lemma, we obtain a map

$$(3.37) \quad \Delta = \text{“Wh}(\alpha) \circ \Gamma_{\tilde{G}}^{-1} \circ \phi(\iota) \circ \omega_H^{-1}\text{”} : \ker(1 \otimes \iota^{ab}) \rightarrow \mathcal{C}.$$

Next we consider the following two subgroups of G

$$\begin{aligned} H_0 &= H \cap [\tilde{G}, \tilde{G}] \\ H_1 &= \left\langle h \in H \mid h = [\tilde{g}_1, \tilde{g}_2] \text{ for } \tilde{g}_1, \tilde{g}_2 \in \tilde{G} \right\rangle \end{aligned}$$

that is to say H_1 is the subgroup of H which is generated by \tilde{G} commutators which lie in H ; so that obviously $[H, H] \subset H_1 \subset H_0$. We write $\tilde{\kappa}_\alpha$ for the composite

$$\tilde{\kappa}_\alpha : \frac{R}{(1-F)R} \otimes H_0 \twoheadrightarrow \ker(1 \otimes \iota^{ab}) \xrightarrow{\Delta} \mathcal{C}.$$

Proposition 3.22. *The map $\tilde{\kappa}_\alpha$ induces an isomorphism*

$$\kappa_\alpha : \frac{R}{(1-F)R} \otimes \frac{H_0}{H_1} \xrightarrow{\sim} \mathcal{C}$$

which is natural with respect to maps between group extensions.

Proof. See Proposition 16 in [O1]. Consider the maps

$$(3.38) \quad \mathrm{SK}_1(R[\tilde{G}]) \xrightarrow{\alpha_{1*}} \mathrm{SK}_1(R[\tilde{G}/H_1]) \xrightarrow{\alpha_{2*}} \mathrm{SK}_1(R[G]).$$

From the proof of Proposition 3.9, because H_1 is generated by commutators, we know that α_{1*} is surjective. By repeated use of the argument used to show Corollary 3.18, we see that α_{2*} is injective, because H/H_1 contains no commutators.

We now first prove the result by taking the case when $H = \langle c \rangle$ is of order p and contains no commutators. First we recall from Lemma 3.7 that

$$\tau : \frac{p\phi((1-c)R[G])}{\mathcal{L}(1+(1-c)R[G])} \xrightarrow{\sim} \frac{R}{(1-F)R + pR}.$$

We know

$$\Gamma(\ker(\mathrm{Wh}(\alpha))) = \nu(\mathrm{Det}(1 + (1-c)R[G]))$$

so that we obtain the exact sequence

$$\ker(\mathrm{Wh}(\alpha)) \xrightarrow{\Gamma} p(1-c)\phi(R[\tilde{G}]) \rightarrow \frac{R}{(1-F)R} \otimes \langle c \rangle \rightarrow 0.$$

We use the exact sequence (3.32) to form the diagram

$$\begin{array}{ccccccccc} 1 & \rightarrow & \mathrm{SK}_1(R[\tilde{G}]) & \rightarrow & \mathrm{Wh}(R[\tilde{G}]) & \rightarrow & \mathrm{Wh}'(R[\tilde{G}]) & \rightarrow & 1 \\ & & \downarrow \alpha_* & & \downarrow \mathrm{Wh}(\alpha) & & \downarrow \mathrm{Wh}'(\alpha) & & \\ 1 & \rightarrow & \mathrm{SK}_1(R[G]) & \rightarrow & \mathrm{Wh}(R[G]) & \rightarrow & \mathrm{Wh}'(R[G]) & \rightarrow & 1 \end{array}$$

and we consider the further diagram

$$(3.39) \quad \begin{array}{ccccccc} \ker \text{Wh}(\alpha) & \xrightarrow{\Gamma} & (1-c)\phi(R[\tilde{G}]) & \xrightarrow{\tau} & R_F \otimes \langle c \rangle & \rightarrow & 0 \\ \downarrow \beta & & \downarrow p & & \downarrow & & \\ 0 \rightarrow \ker \text{Wh}'(\alpha) & \xrightarrow{\varpi} & p(1-c)\phi(R[\tilde{G}]) & \xrightarrow{\omega} & \ker(R_F \otimes \tilde{G}^{\text{ab}} \rightarrow R_F \otimes G^{\text{ab}}) & \rightarrow & 0 \\ \downarrow & & \downarrow & & & & \\ \text{coker}(\alpha_*) & & 0 & & & & \\ \downarrow & & & & & & \\ 0 & & & & & & \end{array}$$

here the left hand vertical column is exact by applying the snake lemma to the previous diagram; the middle horizontal row is exact by applying the snake lemma to the diagram

$$(3.40) \quad \begin{array}{ccccccc} 1 & \rightarrow & \text{Wh}'(R[\tilde{G}]) & \rightarrow & \phi(I_{\tilde{G}}) & \rightarrow & R_F \otimes \tilde{G}^{\text{ab}} \rightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \rightarrow & \text{Wh}'(R[G]) & \rightarrow & \phi(I_G) & \rightarrow & R_F \otimes G^{\text{ab}} \rightarrow 1 \end{array}$$

and the top horizontal row is exact by Lemma 3.7.

Recall that by hypothesis $H = \langle c \rangle$ contains no commutators, so that $H_1 = (1)$; we now consider separately: Case 1, when $H \not\subseteq [G, G]$; and Case 2, when $H \subset [G, G]$.

Case 1. Then $H_0 = H_1 = (1)$; in this case $H = \ker(\tilde{G}^{\text{ab}} \rightarrow G^{\text{ab}})$ and using diagram (3.39) we see that $\ker(\tau) = \ker(\omega)$; hence β is onto and we have shown:

$$H_0/H_1 = (1) = \text{coker}(\alpha_*) = \text{coker}(\text{SK}_1(R[\tilde{G}]) \rightarrow \text{SK}_1(R[G])).$$

Case 2. Then $H = H_0 = \langle c \rangle$ and again $H_1 = (1)$ in this case we have $\tilde{G}^{\text{ab}} \xrightarrow{\sim} G^{\text{ab}}$ and so

$$\text{coker}(\text{SK}_1(R[\tilde{G}]) \rightarrow \text{SK}_1(R[G])) = \text{coker}(\alpha_*) = R_F \otimes \langle c \rangle.$$

(Recall again that R_F stands for the cokernel $R/(1-F)R$.)

Let $H_1 \subset H'_1 \subset \tilde{G}$ with H'_1/H_1 central of order p in \tilde{G}/H_1 ; we then factor $\alpha_2 : \tilde{G}/H_1 \rightarrow G$ as

$$\tilde{G}/H_1 \xrightarrow{\alpha'_2} \tilde{G}/H'_1 \xrightarrow{\beta} G.$$

The result then follows by the above applied to α'_2 and an easy induction theorem on $|G/H_1|$ using the exact sequence

$$1 \rightarrow \text{coker}(\alpha'_{2*}) \rightarrow \text{coker}(\alpha_{2*}) \rightarrow \text{coker}(\beta_*) \rightarrow 1.$$

□

If we write $G = \mathcal{F}/\mathcal{R}$, with \mathcal{F} a free group, then by Hopf's theorem (see Chapter II, Theorem 5.3 and also Exercise 6 in Chapter II of [B]), we have an isomorphism

$$H_2(G, \mathbb{Z}) = \frac{\mathcal{R} \cap [\mathcal{F}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}]}$$

and, given a surjection $\alpha : \tilde{G} \rightarrow G$ with $H = \ker(\alpha)$ contained in the center of \tilde{G} , we get a map

$$\delta^\alpha : H_2(G, \mathbb{Z}) \rightarrow H \quad \text{with } \text{Im}(\delta^\alpha) = H \cap [\tilde{G}, \tilde{G}].$$

From Lemma 17 in [O1] we have the following entirely group theoretic result (in which the ring R plays no role):

Lemma 3.23. *For any p -group G there are central extensions of p -groups*

$$1 \rightarrow H_1 \rightarrow \tilde{G}_1 \xrightarrow{\alpha_1} G \rightarrow 1 \quad \text{and} \quad 1 \rightarrow H_2 \rightarrow \tilde{G}_2 \xrightarrow{\alpha_2} G \rightarrow 1$$

such that: (i) δ^{α_1} is an isomorphism; and (ii) any automorphism of G lifts to an automorphism of \tilde{G}_2 , and for any normal subgroup K of G , if we write $\tilde{K} = \alpha_2^{-1}(K)$, then the sub-extension

$$1 \rightarrow H_2 \rightarrow \tilde{K} \xrightarrow{\alpha} K \rightarrow 1$$

has the property that δ^α is an injection.

In the Introduction we defined the subgroup $H_2^{\text{ab}}(G, \mathbb{Z})$ of $H_2(G, \mathbb{Z})$. Since for an abelian group A we know that $H_2(A, \mathbb{Z}) = A \wedge A$ we see easily that $\delta^\alpha(H_2^{\text{ab}}(G, \mathbb{Z}))$ is equal to the subgroup H generated by $h \in H$ where h is a commutator in \tilde{G} . Note that for any two elements of \tilde{G} whose commutator lies in H , their images in G commute and so lie in an abelian subgroup of G .

We now suppose that

$$1 \rightarrow H \rightarrow \tilde{G} \xrightarrow{\alpha} G \rightarrow 1$$

is an arbitrary extension of finite p -groups, and we again define H_0 and H_1 as above; so that H_0/H_1 is central in \tilde{G}/H_1 . Assembling together the above we have maps

$$\begin{aligned} \text{SK}_1(R[G]) &\rightarrow \text{coker}(\text{SK}_1(R[\tilde{G}]) \rightarrow \text{SK}_1(R[G])) \\ &\xrightarrow{\kappa_\alpha} R_F \otimes H_0/H_1 \xleftarrow{1 \otimes \delta^\alpha} R_F \otimes \overline{H}_2(G, \mathbb{Z}). \end{aligned}$$

Recall here that, as in the Introduction, $\overline{H}_2(G, \mathbb{Z}) = H_2(G, \mathbb{Z})/H_2^{\text{ab}}(G, \mathbb{Z})$.

For any such group extension where δ^α is an injection (for instance when the group extension satisfies condition (i) of Lemma 3.23), δ^α is then obviously an isomorphism; and hence $1 \otimes \delta^\alpha$ is an isomorphism. In these circumstances we denote the composite map as

$$(3.41) \quad \Theta_{R[G]} : \text{SK}_1(R[G]) \rightarrow R_F \otimes \overline{H}_2(G, \mathbb{Z})$$

and we note that by the naturality of the maps involved, $\Theta_{R[G]}$ is in fact independent of the particular choice of extension used (where δ^α is an injection).

Theorem 3.24. *For any p -group G and any ring R which satisfies the Standing Hypotheses the map*

$$\Theta_{R[G]} : \text{SK}_1(R[G]) \rightarrow \frac{R}{(1-F)R} \otimes \overline{H}_2(G, \mathbb{Z}).$$

is an isomorphism.

The proof of Theorem 3.24 is exactly the same as the proof of Theorem 3 in [O1] with R now replacing \mathbb{Z}_p . We highlight the key remaining steps in the proof in order to provide an overview for the reader's convenience. As a first step towards proving the theorem, we note that the proof of Proposition 18 in [O1] now extends to give:

Proposition 3.25. *For any ring R satisfying the Standing Hypotheses, given a surjection of p -groups $\alpha : \tilde{G} \rightarrow G$, if $\Theta_{R[\tilde{G}]}$ is an isomorphism, then $\Theta_{R[G]}$ is also an isomorphism.*

We now quote the following two lemmas (see Lemmas 19 and 20 in [O1]) whose proofs are now also the same as in loc. cit.

Lemma 3.26. *Suppose H is a normal subgroup of the p -group G with the property that G/H is cyclic; then the boundary map*

$$\partial : \ker(\mathrm{Wh}'(R[H]) \rightarrow \mathrm{Wh}'(R[G])) \rightarrow \mathrm{coker}(\mathrm{SK}_1(R[H]) \rightarrow \mathrm{SK}_1(R[G]))$$

is surjective.

Lemma 3.27. *Suppose H is a normal subgroup of the p -group G with the property that G/H is cyclic; then*

$$\mathrm{coker}(\mathrm{SK}_1(R[H]) \rightarrow \mathrm{SK}_1(R[G])) \cong R_F \otimes \mathrm{coker}(\overline{H}_2(H, \mathbb{Z}) \rightarrow \overline{H}_2(G, \mathbb{Z})).$$

In particular, if $\mathrm{SK}_1(R[H]) = \{1\}$, then $\Theta_{R[G]}$ is an isomorphism.

In order to prove Theorem 3.24, we can then piece Proposition 3.25 and Lemmas 3.26 and 3.27 together in an entirely group theoretical way, by studying a suitable category of central extensions of the group G . The details are exactly the same as those for the proof of Theorem 3 in [O1].

4. CHARACTER ACTION AND REDUCTION TO ELEMENTARY GROUPS

In this section we do not need to assume that R supports a lift of Frobenius.

4.a. Character action on SK_1 . Let $G_0(\mathbb{Z}_p[G])$ denote the Grothendieck group of finitely generated $\mathbb{Z}_p[G]$ -modules and let $G_0^{\mathbb{Z}_p}(\mathbb{Z}_p[G])$ denote the Grothendieck group of finitely generated $\mathbb{Z}_p[G]$ -modules which are projective over \mathbb{Z}_p . From 38.42 and 39.9 in [CR2] we have:

Proposition 4.1. *We have*

$$G_0^{\mathbb{Z}_p}(\mathbb{Z}_p[G]) \xrightarrow{\cong} G_0(\mathbb{Z}_p[G]) \xrightarrow{\cong} G_0(\mathbb{Q}_p[G])$$

with the first isomorphism induced by the natural embedding of categories and the second isomorphism induced by the extension of scalars map $\otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Proposition 4.2. *Let S denote an integral domain containing \mathbb{Z}_p . Then $G_0^{\mathbb{Z}_p}(\mathbb{Z}_p[G])$ and hence, by the previous proposition, $G_0(\mathbb{Q}_p[G])$, acts naturally on $K_1(S[G])$ via the following rule: for a $\mathbb{Z}_p[G]$ -lattice L and for an element of $K_1(S[G])$ represented by a*

pair (P, α) (where P is a projective $S[G]$ -module and α is an $S[G]$ -automorphism of P), then L sends (P, α) to the pair

$$(L \otimes P, (1 \otimes \alpha)).$$

Note also for future reference that the functor $G \rightarrow K_1(S[G])$ is a Frobenius module for the Frobenius functor $G \rightarrow G_0^{\mathbb{Z}_p}(\mathbb{Z}_p[G])$ (see page 4 in [CR2] and also [L]). Also, $G \rightarrow SK_1(S[G])$ is a Frobenius submodule of $G \rightarrow K_1(S[G])$ and therefore the action of $G_0(\mathbb{Q}_p[G])$ on $K_1(S[G])$ induces an action on $\text{Det}(\text{GL}(S[G]))$ (see Ullom's Theorem in 2.1 of [T], and see also below for his explicit description of this action).

Proof. See Proposition 5.2 in [CPT1].

4.b. Brauer Induction. Let N^c denote our chosen algebraic closure of the field N and, for a positive integer m , μ_m denotes the group of roots of unity of order m in N^c . We then identify $\text{Gal}(N(\mu_m)/N)$ as a subgroup of $(\mathbb{Z}/m\mathbb{Z})^\times$ in the usual way. Recall that a semi-direct product of a cyclic group C (of order m , say, which is coprime to p) by a p -group P , $C \rtimes P$, is called **N - p -elementary** (see page 112 in [S]) if for given $\pi \in P$ there exists

$$t = t(\pi) \in \text{Gal}(N(\mu_m)/N) \subseteq (\mathbb{Z}/m\mathbb{Z})^\times$$

such that for all $c \in C$, we have $\pi c \pi^{-1} = c^t$. The direct product $C \times P$ is called a **p -elementary group**. We say that a group is **N -elementary** if it is **N - p -elementary** for some p . We will denote by $\mathcal{E}_p(N)$ the class of N - p -elementary groups.

Theorem 4.3. *For a given field N of characteristic zero and for a given finite group G there exists an integer l coprime to p such that*

$$l \cdot G_0(N[G]) \subseteq \sum_J \text{Ind}_J^G(G_0(N[J]))$$

where J ranges over the N - p -elementary subgroups of G .

Proof. See Theorem 28 in [S]. □

4.c. Mackey functors and Green rings. We now briefly introduce the notions of a Mackey functor, a Frobenius functor and a Green ring (see 38.4 in [CR2] and [Bo] for details).

A Frobenius functor consists of a collection of commutative rings $F(G)$, one for each finite group G together with induction and restriction maps of rings for each pair of groups $K \subset H$

$$\text{Ind}_K^H : F(K) \rightarrow F(H), \quad \text{Res}_K^H : F(H) \rightarrow F(K)$$

with the properties:

(1) for $K \subset H \subset J$ we have

$$\text{Ind}_H^J \circ \text{Ind}_K^H = \text{Ind}_K^J, \quad \text{Res}_K^H \circ \text{Res}_H^J = \text{Res}_K^J;$$

(2) for $x \in F(H)$, $y \in F(K)$ we have

$$x \cdot \text{Ind}_K^H(y) = \text{Ind}_K^H(\text{Res}_K^H(x) \cdot y).$$

A Frobenius module N over the Frobenius functor F is a collection of $F(G)$ -modules $N(G)$, one for each finite group G , together with induction and restriction maps of rings for each pair of groups $K \subset H$

$$\text{Ind}_K^H : N(K) \rightarrow N(H), \quad \text{Res}_K^H : N(H) \rightarrow N(K)$$

satisfying the transitivity properties as in (1) above together with the following three properties:

(i) for $x \in F(H)$, $m \in N(H)$

$$\text{Res}_K^H(x) \cdot \text{Res}_K^H(m) = \text{Res}_K^H(x \cdot m);$$

(ii) and for $m' \in N(K)$

$$x \cdot \text{Ind}_K^H(m') = \text{Ind}_K^H(\text{Res}_K^H(x) \cdot m');$$

(iii) and for $x' \in F(K)$

$$\text{Ind}_K^H(x') \cdot m = \text{Ind}_K^H(x' \cdot \text{Res}_K^H(m)).$$

A Mackey functor with values in the category of abelian groups consists of a collection of abelian groups $M(G)$, one for each finite group G , together with induction and restriction maps for each pair of groups $K \subset H$

$$\text{Ind}_K^H : M(K) \rightarrow M(H), \quad \text{Res}_K^H : M(H) \rightarrow M(K)$$

and conjugation maps $c_{x,H} : M(H) \rightarrow M({}^xH)$ for $x \in G$, $H \subset G$ where we write ${}^xH = xHx^{-1}$, $H^x = x^{-1}Hx$. These maps are then required to satisfy the following properties:

1. For $L \subset K \subset H$ we have $\text{Ind}_K^H \circ \text{Ind}_L^K = \text{Ind}_L^H$ and $\text{Res}_L^K \circ \text{Res}_K^H = \text{Res}_L^H$.
2. For $x, y \in G$ and $H \subset G$ we have $c_{y, {}^xH} \circ c_{x,H} = c_{yx,H}$.
3. For $x \in G$ and $K \subset H \subset G$ we have

$$c_{x,H} \circ \text{Ind}_K^H = \text{Ind}_{{}^xK}^{{}^xH} \circ c_{x,K} \quad \text{and} \quad c_{x,H} \circ \text{Res}_K^H = \text{Res}_{{}^xK}^{{}^xH} \circ c_{x,K}.$$

4. For $x \in H$ we have $c_{x,H} = \text{id}_H$.
5. (The Mackey axiom.) For $L \subset H \supset K$ we have

$$\text{Res}_L^H \circ \text{Ind}_K^H = \sum_{x \in L \backslash H/K} \text{Ind}_{L \cap {}^xK}^L \circ c_{x, L \cap {}^xK} \circ \text{Res}_{L \cap {}^xK}^K.$$

A Green ring is a commutative ring-valued Mackey functor \mathcal{R} which is a Frobenius functor; and a Green module for the Green ring \mathcal{R} is a Frobenius module over \mathcal{R} which is also a Mackey functor. (See [D] and page 246 in [O5] for details.)

Examples. (i) For a given prime p , the functor $G \rightarrow G_0(\mathbb{Z}_p[G])$ is a Green ring and the functors $G \rightarrow K_1(R[G])$, $\text{Det}(K_1(R[G]))$, $\text{SK}_1(R[G])$ are Green modules for the Green ring $G \rightarrow G_0(\mathbb{Z}_p[G])$.

(ii) For a given prime p , the functor $G \rightarrow H^0(G, \mathbb{Z}_p)$ is a Green ring and the functors $G \rightarrow H_2(G, \mathbb{Z}_p)$, $H_2^{ab}(G, \mathbb{Z}_p)$, $\overline{H}_2(G, \mathbb{Z}_p)$ are Green modules for this Green ring. (See page 281 in [O5].)

Let \mathcal{C} be an arbitrary class of finite groups which is closed with respect to subgroups and isomorphic images. For a given finite group G we write $\mathcal{C}(G)$ for the set of subgroups of G which belong to the class \mathcal{C} .

We say that the Mackey functor M is **generated** by \mathcal{C} if for any G

$$\bigoplus_{H \in \mathcal{C}(G)} M(H) \xrightarrow{\Sigma \text{Ind}_H^G} M(G)$$

is surjective.

We say that the Mackey functor M is **\mathcal{C} -computable with respect to induction** if for any G

$$M(G) \cong \varinjlim_{H \in \mathcal{C}(G)} M(H)$$

where the direct limit is taken with respect to the induction and conjugation maps.

We say that the Mackey functor M is **\mathcal{C} -computable with respect to restriction** if for any G

$$M(G) \cong \varprojlim_{H \in \mathcal{C}(G)} M(H)$$

where the inverse limit is taken with respect to the restriction and conjugation maps.

From Theorem 11.1 in [O5] we have:

Theorem 4.4. (*Dress*) *Let \mathcal{R} be a Green ring, M be a Green module for \mathcal{R} and let \mathcal{C} be a class of finite groups with the property that \mathcal{R} is \mathcal{C} -generated; then M is \mathcal{C} -computable with respect to both induction and restriction.*

Definition 4.5. *A Mackey functor is called **p -local** if it is a \mathbb{Z}_p -module valued functor.*

Using Brauer induction as in 4.b above it can also be shown that (see Theorem 11.2 in [O1]):

Theorem 4.6. *Let \mathcal{E} , resp. \mathcal{E}_p , denote the class of \mathbb{Q} -elementary, resp. of \mathbb{Q}_p - p -elementary, groups.*

(i) *As previously, suppose that R has field of fractions N and let X be an additive functor from the category of R -orders in semi-simple N -algebras with bimodule morphisms to the category of abelian groups. Then $M(G) = X(R[G])$ is a Mackey functor and is in fact a Green module for the Green ring $G \rightarrow G_0(\mathbb{Z}[G])$ and M is \mathcal{E} -computable with respect to both induction and restriction.*

(ii) *Suppose further that X is p -local; then $M(G) = X(R[G])$ is a Green module for the Green ring $G \rightarrow G_0(\mathbb{Z}_p[G])$ and M is \mathcal{E}_p -computable with respect to both induction and restriction.*

From Theorem 11.9 in [O5] we have:

Theorem 4.7. *Let p denote a chosen prime number, let R be a p -adically complete integrally closed integral domain of characteristic zero with field of fractions N and let X be a p -local additive functor on the category of R -orders with bimodule morphisms.*

For any positive integer n which is prime to p , let ζ_n denote a primitive n -th root of unity in \mathbb{Q}_p^c and set $R[\zeta_n] = R \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_n]$. Then:

(i) $X(R[G])$ is computable with respect to induction from p -elementary groups if, and only if, for any n as above, any p -group G and any homomorphism $t : G \rightarrow \text{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p)$ with $H = \ker(t)$, the induction map

$$\text{Ind}_H^G : H_0(G/H, X(R[\zeta_n][H])) \rightarrow X(R[\zeta_n] \circ G)$$

is bijective.

(ii) $X(R[G])$ is computable with respect to restriction to p -elementary groups if, and only if, for any n , as above, any p -group p and any homomorphism $t : G \rightarrow \text{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p)$ with $H = \ker(t)$, the restriction map

$$\text{Res}_H^G : X(R[\zeta_n] \circ G) \rightarrow H^0(G/H, X(R[\zeta_n][H]))$$

is bijective.

To obtain this theorem from Theorem 11.9 in [O5] it is helpful to note that (see Sect. 6 in [CPT1]) we can decompose the ring $R[\zeta_n]$ into a product of integral domains.

5. SK₁ FOR p -ELEMENTARY GROUPS

In this section we assume that in addition R is a normal ring. Here G is a \mathbb{Q}_p - p -elementary group, written $G = P \rtimes C$ as above, with P a p -group and C a cyclic group of order prime to p . We therefore have decompositions of algebras:

$$\begin{aligned} \mathbb{Z}_p[C] &= \prod_m \mathbb{Z}_p[m] \quad \text{where} \quad \mathbb{Z}_p[m] = \mathbb{Z}[\zeta_m] \otimes_{\mathbb{Z}} \mathbb{Z}_p \\ R[C] &= \prod_m R[m] \quad \text{where} \quad R[m] = \mathbb{Z}_p[m] \otimes_{\mathbb{Z}_p} R \end{aligned}$$

and where m runs over the divisors of the order of C .

We fix a surjective abelian character $\chi : C \rightarrow \langle \zeta_m \rangle$. We let A_m denote the image of P in $\text{Aut}\langle \zeta_m \rangle$ given by conjugation; thus, by the definition of a \mathbb{Q}_p - p -elementary group, A_m identifies as a subgroup of the cyclic group $\text{Gal}(\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p)$ and we let H_m denote the kernel of the map from P to A_m . We endow $R[m]$ with the lift of Frobenius given by the tensor product of the lift of Frobenius on R and the Frobenius automorphism of $\mathbb{Z}_p[m]$. From Lemma 6.1 in [CPT1], we know that each $R[m]$ decomposes as a product of integral domains each of which satisfies the Standing Hypotheses.

Recall that each twisted group ring $R[m] \circ P$ contains the standard group ring $R[m][H_m]$, and, as per 6.a in [CPT1], we have the induction and restriction maps

$$K_1(R[m][H_m]) \xrightleftharpoons[i_*]{r_\chi} K_1(R[m] \circ P).$$

In most of the remaining of this section we fix m and drop the index m where possible; so, in particular, we write H, A, r for H_m, A_m, r_m and set $B = R[m]$. We write I_H for the augmentation ideal $I(B[H])$ and we let I_P denote the two-sided $B \circ P$ ideal generated by I_H . Note for future reference that, since $R[P]I_H$ is contained in the Jacobson radical of $R[P]$ and since B commutes with H , by the definition of H ,

it follows that I_P is contained in the Jacobson radical of $B \circ P$. In the usual way we have the relative K-groups $K_1(B[H], I_H)$, and $K_1(B \circ P, I_P)$.

Recall that from Theorem 6.2 and Proposition 6.3 in [CPT1] we have

Theorem 5.1. *We have*

$$r(\text{Det}((B \circ P)^\times)) = \text{Det}(B[H]^\times)^A \supseteq r \circ i_*(\text{Det}(B[H]^\times)).$$

Using the decomposition $\text{Det}(B[H]^\times) = \text{Det}(1 + I_H) \times B^\times$, we obtain inclusions

$$(5.1) \quad r(\text{Det}(1 + I_P)) = \text{Det}(1 + I_H)^A \supseteq r \circ i_*(\text{Det}(1 + I_H))$$

and

$$\text{Det}(1 + I_P) = i_*(\text{Det}(1 + I_H)).$$

We shall now generalize a result of R. Oliver by establishing a corresponding result for K'_1 by showing:

Proposition 5.2. *We have $i_*(K'_1(B[H], I_H)) = K'_1(B \circ P, I_P)$.*

Proof. Note that the result is immediate if $H = \{1\}$ and so we may henceforth assume that H contains an element c of order p which is central in P . We write $t : P/H \rightarrow A$ for the isomorphism induced by conjugation. We let c denote a central element of order p which lies in H ; note that here we are not necessarily assuming that c is a commutator in H . We let $\overline{H} = H/\langle c \rangle$, $\overline{P} = P/\langle c \rangle$ and we write $I_{\overline{H}}$ for $I(B[\overline{H}])$ and $I_{\overline{P}}$ for $I(B \circ \overline{P})$. Our proof proceeds in two steps.

Step 1. With the above notation we consider the diagram with exact rows, where the downward arrows are induced by i

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{K}_1 & \xrightarrow{\text{defn}} & K'_1(B[H], I_H) & \rightarrow & K'_1(B[\overline{H}], I_{\overline{H}}) \rightarrow 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \\ 0 & \rightarrow & \mathcal{K}_2 & \xrightarrow{\text{defn}} & K'_1(B \circ P, I_P) & \rightarrow & K'_1(B \circ \overline{P}, I_{\overline{P}}) \rightarrow 0 \end{array}$$

and recall the exact sequence

$$0 \rightarrow (1 - c)B[H] \rightarrow I_H \rightarrow I_{\overline{H}} \rightarrow 0.$$

We rewrite the top row of above diagram as

$$\begin{array}{ccccccc} & & & 1 & \rightarrow & 1 & \\ & & & \uparrow & & \uparrow & \\ 1 & \rightarrow & \mathcal{K}_1 & \rightarrow & K'_1(B[H], I_H) & \rightarrow & K'_1(B[\overline{H}], I_{\overline{H}}) \rightarrow 1 \\ & & \uparrow \sigma & & \uparrow \sigma' & & \uparrow \\ 1 & \rightarrow & \text{GL}(B[H], (1 - c)) & \rightarrow & \text{GL}(B[H], I_H) & \rightarrow & \text{GL}(B[\overline{H}], I_{\overline{H}}) \rightarrow 1 \end{array}$$

We claim that σ is surjective: to see this, first note that the map $\ker(\sigma) \rightarrow \ker(\sigma')$ induced by the map $\text{GL}(B[H], I_H) \rightarrow \text{GL}(B[\overline{H}], I_{\overline{H}})$ is

$$\rho : \text{GL}(B[H], I_H) \cap E(B[H]) \rightarrow \text{GL}(B[\overline{H}], I_{\overline{H}}) \cap E(B[\overline{H}]).$$

By the snake lemma, it will suffice to show that the map ρ is surjective; to see this, given $\bar{e} = \bar{e}_1 \cdots \bar{e}_n \in \text{GL}(B[\overline{H}], I_{\overline{H}}) \cap E(B[\overline{H}])$, we choose $e_i \in E(B[H])$ with image

$\bar{e}_i \in E(B[\bar{H}])$ and set $e = e_1 \cdots e_n$; then, as the augmentation map ε_H factors through $\varepsilon_{\bar{H}}$, we see that $\varepsilon_H(e) = 1$ because $\varepsilon_{\bar{H}}(\bar{e}) = 1$ and so $e \in \text{GL}(B[H], I_H)$. A similar argument shows that $\text{GL}(B \circ P, (1 - c)I_P)$ maps onto \mathcal{K}_2 and we have shown:

$$(5.2) \quad \text{GL}(B[H], (1 - c)) \twoheadrightarrow \mathcal{K}_1, \quad \text{GL}(B \circ P, (1 - c)) \twoheadrightarrow \mathcal{K}_2.$$

We now show that β is surjective by arguing by induction on the order of the group H . Note that if $\bar{H} = \{1\}$, then $I_{\bar{H}} = 0 = I_{\bar{P}}$, and so in this case we trivially have

$$K'_1(B[H], \bar{I}) = 0 = K'_1(B \circ P, \bar{I}_P)$$

and this starts the induction. By the inductive hypothesis we may assume that γ is surjective. Therefore, by the snake lemma, it will now suffice to show that α is surjective.

Step 2. As $(1 - c)B[H]$ resp. $(1 - c)B \circ P$ lies in the radical of $B[H]$ resp. $B \circ P$, we know from Lemma 2.2 (b) that there are surjections

$$(5.3) \quad \kappa_H : (1 + (1 - c))B[H] \twoheadrightarrow \mathcal{K}_1, \quad \kappa_P : (1 + (1 - c)B \circ P) \twoheadrightarrow \mathcal{K}_2$$

where we abbreviate κ_{I_H} to κ_H and κ_{I_P} to κ_P . In order to show that the map α is surjective, we shall show that for each $k \geq 1$ the map

$$i_{*k} : \frac{\kappa_H(1 + (1 - c)^k B[H])}{\kappa_H(1 + (1 - c)^{k+1} B[H])} \rightarrow \frac{\kappa_P(1 + (1 - c)^k B \circ P)}{\kappa_P(1 + (1 - c)^{k+1} B \circ P)}$$

is surjective. This will then prove the proposition because the $(1 - c)$ -adic and p -adic topologies are cofinal in both $(1 - c)B[H]$ and $(1 - c)B \circ P$.

Recall that $B = R[m]$ and $A \subset \text{Gal}(R[m]/R)$. Let $\bar{B} = B/pB$, $\bar{R} = R/pR$ and recall that by the Standing Hypothesis \bar{R} is an integral domain and $\bar{B} = \bar{R} \otimes \mathbb{F}_p[m]$. To establish the surjectivity of i_{*k} , we choose a normal basis generator $\tilde{\mu}$ of $\mathbb{F}_p[m]$ over \mathbb{F}_p as follows: let ζ denote a primitive m -th root of unity in the algebraic closure of \mathbb{F}_p ; then A identifies as a subgroup of $\text{Gal}(\mathbb{F}_p(\zeta)/\mathbb{F}_p)$ and $\mathbb{F}_p[m]$ identifies as a product of copies of $\mathbb{F}_p(\zeta)$. We then take $\tilde{\mu}$ to be the direct product of copies of a normal integral basis of $\mathbb{F}_p(\zeta)/\mathbb{F}_p(\zeta)^A$ and so we observe that $\tilde{\mu}$ is an invertible element of $\mathbb{F}_p[m]$. This then affords a normal basis generator (also denoted $\tilde{\mu}$) of \bar{B} over the group ring $\bar{B}^A[A]$. We then choose a lift $\mu \in B$ of $\tilde{\mu}$. Then for $g \in P - H$, $g(\tilde{\mu}) \neq \tilde{\mu}$ and so $\mu^{-1}(g(\mu)) - 1 \in B^\times$. So for any $\lambda \in B$ we can find λ' such that

$$\lambda = \lambda'(\mu^{-1}(g(\mu)) - 1)$$

and we note that $\text{coker}(i_{*k})$ is generated by elements of the form $1 + (1 - c)^k \lambda g^{-1}$ with $g \in P - H$.

Using the identity $g(\mu) = g \cdot \mu \cdot g^{-1}$, we then note the identity

$$\begin{aligned} \kappa_P(1 + (1 - c)^k \lambda g) &= \kappa_P(1 + (1 - c)^k \lambda'(\mu^{-1}(g(\mu)) - 1)g) \\ &= \kappa_P(1 + \mu^{-1}(1 - c)^k \lambda' g \mu - (1 - c)^k \lambda' g) \\ &\equiv \kappa_P((1 + (1 - c)^k \mu^{-1} \lambda' g \mu)(1 - (1 - c)^k \lambda' g)) \pmod{(1 - c)^{k+1}}. \end{aligned}$$

Using the fact that, because $c \in H$, it commutes with B , we have shown

$$\kappa_P(1 + (1-c)^k \lambda g^{-1}) \equiv \kappa_P(\mu^{-1}(1 + (1-c)^k \lambda' g^{-1}) \mu(1 + (1-c)^k \lambda' g^{-1})^{-1}) \pmod{(1-c)^{k+1}}$$

which, being a commutator, has trivial image in $K'_1(B \circ P, I_P)$. \square

Recall we write I_H for $I_{B[H]}$.

Proposition 5.3. *$\text{Det}(1 + I_H)$ is a cohomologically trivial A -module; hence*

$$\text{Det}(1 + I_H)^A = H^0(A, \text{Det}(1 + I_H)) = H_0(A, \text{Det}(1 + I_H))$$

and

$$H_1(A, \text{Det}(1 + I_H)) = \{1\} = \widehat{H}^{-1}(A, \text{Det}(1 + I_H)) = \{1\}.$$

Proof. Recall that we write $\mathcal{A}_H = \ker(B[H] \rightarrow B[H^{\text{ab}}])$. With the notation of 3.a (using the fact that B is a direct product of algebras which satisfy the Standing Hypotheses), we have an exact sequence

$$0 \rightarrow \phi(\mathcal{A}_H) \rightarrow \text{Det}(1 + I_H) \rightarrow 1 + I(B[H^{\text{ab}}]) \rightarrow 1.$$

As seen previously, the B -lattice $\phi(\mathcal{A}_H)$ is $R[A]$ -free and hence is a cohomologically trivial A -module. On the other hand $1 + I(B[H^{\text{ab}}])$ can be filtered with subquotients all of which are isomorphic to \overline{B} . The result then follows since all these terms are also all cohomologically trivial A -modules. \square

Corollary 5.4. *We have $i_*(\text{SK}_1(B[H])) = \text{SK}_1(B \circ P, I_P) = \text{SK}_1(B \circ P)$.*

Proof. By Theorem 1.1 we know that $\text{Det}(K_1(R[G])) = \text{Det}(R[G]^\times)$. Consider the augmentation map $B[H] \rightarrow B$ and the induced map

$$B \circ P \rightarrow B \circ A \cong M_{|A|}(B^A)$$

with the latter isomorphism coming from the proof of Theorem 6.2 in [CPT1].

This then leads us to consider the commutative diagram with exact rows, where the downward arrows are all induced by i_* :

$$(5.4) \quad \begin{array}{ccccccc} 0 & \rightarrow & \text{SK}_1(B[H]) & \rightarrow & K'_1(B[H], I_H) & \rightarrow & \text{Det}(1 + I_H) \rightarrow 1 \\ & & \downarrow \rho & & \downarrow i & & \downarrow \tau \\ 0 & \rightarrow & \text{SK}_1(B \circ P, I_P) & \rightarrow & K'_1(B \circ P, I_P) & \rightarrow & \text{Det}(1 + I_P) \rightarrow 1. \end{array}$$

By Proposition 5.2 we know that the middle downward arrow i is surjective. In order to show that ρ is surjective, by the snake lemma, it will suffice to show that $\ker i$ maps onto $\ker \tau$. Suppose therefore that $\text{Det}(x) \in \ker \tau$. We have seen that the restriction map r is injective and $r \circ i_*$ coincides with the norm map N_A ; so by Proposition 5.3 we deduce that we can write

$$\text{Det}(x) = \prod_{a \in A} \text{Det}(y_a)^{a-1}$$

with $y_a \in 1 + I_H$. Now consider the image of the element $z = \prod y_a^{a-1} \in K'_1(B[H], I_H)$ in $K_1(B \circ P)$: clearly this maps to $\text{Det}(x)$ under Det ; moreover, each term $y_a^{a-1} = a y_a a^{-1} y_a^{-1}$ becomes a commutator in $(B \circ P)^\times$ and so vanishes in $K_1(B \circ P)$, as

required. To conclude we show that $\mathrm{SK}_1(B \circ P) = \mathrm{SK}_1(B \circ P, I_P)$ and this follows from the exact sequence

$$1 \rightarrow I_P \rightarrow B \circ P \rightarrow B \circ A_m \rightarrow 1$$

and the equalities

$$\mathrm{SK}_1(B \circ A_m) = \mathrm{SK}_1(M_{|A_m|}(B^{A_m})) = \mathrm{SK}_1(B^{A_m}) = \mathrm{SK}_1(R \otimes \mathbb{Z}[m]) = \{1\}.$$

□

We conclude this section by showing:

Theorem 5.5. *We have*

$$i_*(\mathrm{SK}_1(B[H])) = H_0(A, \mathrm{SK}_1(B[H]))$$

in $\mathrm{SK}_1(B \circ P)$.

Proof. We start by noting as previously that for $x \in K'_1(B \circ P, I_P)$ and $a \in A$ we have $x^a x^{-1} = 1$ and so $I_A K'_1(B \circ P, I_P) = 0$. Next we note that by taking A -homology of the top exact row in (5.4) and using the above Proposition 5.3 we get

$$\begin{array}{ccccc} H_0(A, \mathrm{SK}_1(B[H])) & \hookrightarrow & H_0(A, K'_1(B[H], I_H)) & \twoheadrightarrow & H_0(A, \mathrm{Det}(1 + I(B[H]))) \\ \downarrow \rho & & \downarrow i & & \downarrow \tau \\ \mathrm{SK}_1(B \circ P, I_P) & \hookrightarrow & K'_1(B \circ P, I_P) & \twoheadrightarrow & \mathrm{Det}(1 + I(B \circ P)) \end{array}$$

Now from the above we know that τ is an isomorphism; by Corollary 5.4 ρ is surjective; by Proposition 5.2 i is surjective; hence, by the snake lemma, it will now suffice to show that i is an isomorphism.

By Lemma 8.3.ii and Lemma 8.9 in [O5] (see also the discussion on page 278 of [O5]) we know that we can find a finite p -group \tilde{H} with a central subgroup Σ so that we have a diagram

$$\begin{array}{ccccccccc} 1 & \rightarrow & \Sigma & \rightarrow & \tilde{P} & \rightarrow & P & \rightarrow & 1 \\ & & \uparrow = & & \uparrow & & \uparrow & & \\ 1 & \rightarrow & \Sigma & \rightarrow & \tilde{H} & \xrightarrow{\alpha_0} & H & \rightarrow & 1 \end{array}$$

with $H_2(\alpha_0) = 0$, where $H_2(\alpha_0)$ is the map $H_2(\tilde{H}, \mathbb{Z}_p) \rightarrow H_2(H, \mathbb{Z}_p)$ induced by α_0 ; hence by Lemma 8.9 in [O5] we know that we have $\mathrm{SK}_1(B[H]) = (1)$; and hence

$$(5.5) \quad K'_1(B[H], I_H) = \mathrm{Det}(1 + I(B[H]));$$

and therefore, by Proposition 5.3, $K'_1(B[H], I_H)$ is A -cohomologically trivial.

To conclude we consider the exact sequence which defines \mathcal{J}

$$1 \rightarrow \mathcal{J} \rightarrow B[\tilde{H}] \rightarrow B[H] \rightarrow 1$$

together with the diagram

$$\begin{array}{ccccccc} K'_1(B[\tilde{H}], \mathcal{J}) & \rightarrow & H_0(A, K'_1(B[\tilde{H}], I_{\tilde{H}})) & \rightarrow & H_0(A, K'_1(B[H], I_H)) & \rightarrow & 1 \\ \downarrow a & & \downarrow b & & \downarrow c & & \\ 1 \rightarrow K'_1(B \circ \tilde{P}, \mathcal{J}) & \rightarrow & K'_1(B \circ \tilde{P}, I_{\tilde{P}}) & \rightarrow & K'_1(B \circ P, I_P) & \rightarrow & 1 \end{array}$$

where the top row is obtained by using the map $K'_1(B[\tilde{H}], \mathcal{J}) \rightarrow H_0(A, K'_1(B[\tilde{H}], \mathcal{J}))$ by taking the A -homology of the exact sequence

$$1 \rightarrow K'_1(B[\tilde{H}], \mathcal{J}) \rightarrow K'_1(B[\tilde{H}], I_{\tilde{H}}) \rightarrow K'_1(B[H], I_H) \rightarrow 1$$

which follows from the snake lemma applied to the diagram

$$\begin{array}{ccccccc} & & K_1(B[\tilde{H}]) & = & K_1(B[\tilde{H}]) & & \\ & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & K'_1(B[\tilde{H}]/\mathcal{J}, I_{\tilde{H}}/\mathcal{J}) & \rightarrow & K_1(B[\tilde{H}]/\mathcal{J}) & \rightarrow & K_1(B[\tilde{H}]/I_{\tilde{H}}) \rightarrow 1. \end{array}$$

The map a is induced by i and so is surjective; the map b is an isomorphism by (5.5) and Theorem 5.1; hence the map c is an isomorphism as required. \square

6. ARBITRARY FINITE GROUPS.

Throughout this section we shall always suppose that R satisfies the Standing Hypotheses and is normal and that G is an arbitrary finite group.

We start by proving Theorem 1.7: we do this by using the induction theorems of Section 4, using the isomorphism $\Theta_{R[G]}$ for p -groups, and the work in Section 5 for \mathbb{Q}_p - p -elementary groups, in order to produce an isomorphism $\Theta_{R[G]}$ for arbitrary finite groups G .

In subsection 6.c we construct Adams operations on the determinantal groups $\text{Det}(R[G]^\times)$. We then use these Adams operations to construct a group logarithm v_G , which naturally extends the group logarithm for p -groups described in 3.a. This enables us to construct a long exact sequence (see Proposition 6.20) which generalizes the exact sequence (1.3) (which was valid only for p -groups); this result provides a deep understanding of the determinantal groups $\text{Det}(R[G]^\times)$. *Inter alia* this sequence, together with other constructions, allows us to provide a more constructive definition of the map $\Theta_{R[G]}$.

6.a. Proof of Theorem 1.7. In this subsection we suppose that R satisfies the Standing Hypotheses. In 3.c we constructed the map $\Theta_{R[G]}$ in the case when G is a p -group. Indeed, in that case, by Theorem 3.24 we have the natural isomorphism

$$(6.1) \quad \Theta_{R[G]} : \text{SK}_1(R[G]) \rightarrow \overline{H}_2(G, R)_\Psi.$$

Suppose now that $G = C \rtimes P$ is a \mathbb{Q}_p - p -elementary group and, with the notation of Section 5, we then have decompositions

$$R[G] = \oplus_m R[H_m] \circ P, \quad R[G_r] = R[C] = \oplus_m R[m]$$

(so that $H_m = \ker(\text{conj} : P \rightarrow \text{Aut}\langle \zeta_m \rangle)$ and $A_m = \text{Im}(\text{conj} : P \rightarrow \text{Aut}\langle \zeta_m \rangle)$). Thus we obtain the further decompositions

$$\begin{aligned} H_2(G, R[G_r]) &= \oplus_m H_2(G, R[m]) = \oplus_m H_2(H_m, R[m]^{A_m}) \\ &= \oplus_m H_2(H_m, R[m]_{A_m}) = \oplus_m H_0(A_m, H_2(H_m, R[m])) \end{aligned}$$

by using the fact that $R[m]$ is A_m -free. We then have similar decompositions for $H_2^{\text{ab}}(G, R[G_r])$, $\overline{H}_2(G, R[G_r])$, $\overline{H}_2(G, R[G_r])_\Psi$.

By Theorem 5.5 there is a natural isomorphism

$$\text{SK}_1(R[m][G]) = \text{SK}_1(\oplus_m R[m][H_m] \circ P) = \oplus_m H_0(A_m, \text{SK}_1(R[m][H_m]))$$

and so applying the functor $H_0(A_m, -)$ to the isomorphism

$$\Theta_{R[m][H_m]} : \text{SK}_1(R[m][H_m]) \rightarrow \overline{H}_2(H_m, R[m])_\Psi$$

we get the isomorphisms for each m

$$H_0(A_m, \Theta_{R[m][H_m]}) : H_0(A_m, \text{SK}_1(R[m][H_m])) \rightarrow H_0(A_m, \overline{H}_2(H_m, R[m])_\Psi).$$

For a \mathbb{Q}_p - p -elementary group G these isomorphisms add together to give the desired isomorphism

$$(6.2) \quad \Theta_{R[G]} : \text{SK}_1(R[G]) \rightarrow \overline{H}_2(G, R[G_r])_\Psi.$$

In conclusion we note that for arbitrary G by Theorem 4.7 (i)

$$\text{SK}_1(R[G]) = \varinjlim_{H \in \mathcal{E}_p(G)} \text{SK}_1(R[H]).$$

To complete the proof of Theorem 1.7 we show:

Lemma 6.1. *We have*

$$\overline{H}_2(G, R[G_r]) = \varinjlim_{H \in \mathcal{E}_p(G)} \overline{H}_2(H, R[H_r]).$$

Proof. Let $E_p(G)$ denote the set of subgroups of G which are p -elementary, so that of course $E_p(G) \subset \mathcal{E}_p(G)$. We start by showing

$$H_2(G, R[G_r]) = \varinjlim_{H \in E_p(G)} H_2(H, R[H_r]);$$

then, since every p -elementary subgroup of G is trivially \mathbb{Q}_p - p -elementary, by Theorem 4.4 it will follow that

$$H_2(G, R[G_r]) = \varinjlim_{H \in \mathcal{E}_p(G)} H_2(H, R[H_r]).$$

Let G_p denote a p -Sylow subgroup of G . Since the index $(G : G_p)$ is a unit of R , we have the equality

$$\text{Cor}_{G_p}^G(H_2(G_p, R[G_r])) = H_2(G, R[G_r]).$$

Next we decompose G_r into disjoint cycles under G_p -conjugation

$$G_r = \bigcup_{i \in I} g_i^{G_p},$$

we let H_i denote the subgroup of G_p that centralizes g_i . Then it follows that

$$H_2(G_p, R[G_r]) = \sum_i H_2(G_p, \sum_{\gamma \in H_i \setminus G_p} Rg_i^\gamma) = \sum_i H_2(H_i, Rg_i)$$

and the result follows because $\langle g_i \rangle \times H_i$ is p -elementary.

The proof that

$$H_2^{\text{ab}}(G, R[G_r]) = \varinjlim_{H \in \mathcal{E}_p(G)} H_2^{\text{ab}}(H, R[H_r])$$

is very similar (see the proof of Corollary 1.8 in Appendix B). \square

6.b. Extending the group logarithm. In this subsection we consider the extension of the group logarithm in the trivial case when $G = \{1\}$ and define

$$\mathcal{L}_R(R^\times) = \left(\log \circ \frac{p}{F} \right) (R^\times) \subset \log(1 + pR) \subset pR$$

where abusively p/F denotes the map on R^\times given by $p/F(u) = u^p/F(u)$.

We define

$$\mathcal{M}(R, F) = \{u \in R^\times \mid F(u) = u^p\}, \quad \Lambda(R, F) = R^\times / \mathcal{M}(R, F).$$

When F is fixed we shall write $\Lambda(R)$ and $\mathcal{M}(R)$ in place of $\Lambda(R, F)$ and $\mathcal{M}(R, F)$. We claim that the following sequence is exact:

$$(6.3) \quad R^\times \xrightarrow{\frac{1}{p}\mathcal{L}_R \oplus \theta} R \oplus \Lambda(R) \xrightarrow{T} R \rightarrow 0$$

where $\theta(u) = u \bmod \mathcal{M}(R)$ and where $T(x \oplus y) = x - \frac{1}{p}\mathcal{L}_R(y)$. Indeed: T is surjective, since obviously R maps onto R ; clearly

$$T\left(\left(\frac{1}{p}\mathcal{L}_R \oplus \theta\right)(u)\right) = \frac{1}{p}\mathcal{L}_R(u) - \frac{1}{p}\mathcal{L}_R(u) = 0;$$

and, if $T(x \oplus y) = 0$, then $x = \frac{1}{p}\mathcal{L}_R(y)$, and so $(\frac{1}{p}\mathcal{L}_R \oplus \theta)(y) = x \oplus y$.

Lemma 6.2. (a) $\mathcal{M}(R) \cap (1 + pR) = \{1\}$.

$$(b) \ker(\mathcal{L}_R) = \begin{cases} \mathcal{M}(R) & \text{if } p > 2, \\ \mathcal{M}(R) \oplus \langle \pm 1 \rangle & \text{if } p = 2. \end{cases}$$

Proof. (a) For the sake of contradiction we consider $1 + p^n x \in \mathcal{M}(R)$ with $n > 0$ and $x \notin pR$. Then we have congruences modulo $p^{n+1}R$

$$1 + p^n x^p \equiv 1 + p^n F(x) \equiv F(1 + p^n x) \equiv (1 + p^n x)^p \equiv 1$$

which implies $x \in pR$.

For (b) we consider the factorization of \mathcal{L}_R given by

$$R^\times \xrightarrow{p/F} 1 + pR \xrightarrow{\log} pR$$

where $p/F(u) = u^p F(u)^{-1}$. The result then follows since, because pR is a prime ideal, we have:

$$\ker(\log : 1 + pR \rightarrow pR) = \begin{cases} \{1\} & \text{if } p > 2, \\ \langle \pm 1 \rangle & \text{if } p = 2. \end{cases}$$

\square

Example 6.3. We can use the structure of R^\times to work out $\mathcal{M}(R)$ and $\Lambda(R)$ in the following cases:

1. If W denotes the valuation ring of a finite non-ramified extension of \mathbb{Q}_p ; then $\mathcal{M}(W) = \mu'_W$, the group of roots of unity of W of order prime to p and $\Lambda(W) \cong pW$.
2. Using $k[t]^\times = k^\times$ we can see that $\mathcal{M}(W\langle t \rangle) = \mu'_W$ and $\Lambda(W\langle t \rangle) \cong 1 + pW\langle t \rangle$.
3. Also $\mathcal{M}(W[[t]]) = \mu'_W$ and $\Lambda(W[[t]]) \cong 1 + (p, t)W[[t]]$.
4. Any unit u of the d.v.r $W\{\{t\}\}$ can be written as $u = t^n v(1 + pr)$ with $r \in W\{\{t\}\}$, $n \in \mathbb{Z}$ and $v \in W[[t]]^\times$; therefore $\mathcal{M}(W\{\{t\}\}) = \mu'_W \times t^\mathbb{Z}$ and we have an exact sequence

$$1 \rightarrow 1 + pW\{\{t\}\} \rightarrow \Lambda(W\{\{t\}\}) \rightarrow 1 + tk[[t]] \rightarrow 1$$

where we write k for the residue class field of W .

For a p -group G we splice the above exact sequence (6.3) together with the exact sequence

$$\text{Det}(1 + I_{R[G]}) \xrightarrow{\nu_G} p\phi(I_{R[G]}) \rightarrow G^{\text{ab}} \otimes \frac{R}{(1-F)R} \rightarrow 0$$

from Theorem 3.17 in [CPT1], and we get the exact sequence for the whole group $\text{Det}R[G]^\times$:

$$\begin{aligned} (6.4) \quad \text{Det}(R[G]^\times) &= \text{Det}(1 + I_{R[G]}) \oplus R^\times \xrightarrow{\nu'_G \oplus \frac{1}{p}\mathcal{L}_R \oplus \theta_G} \phi(I_{R[G]}) \oplus R \oplus \Lambda(R) \\ &= pR[C_G] \oplus \Lambda(R) \rightarrow G^{\text{ab}} \otimes \frac{R}{(1-F)R} \oplus R \end{aligned}$$

where for brevity we set $\nu'_G = \frac{1}{p}\nu_G$ and θ_G is the composition of augmentation and reduction modulo $\mathcal{M}(R)$.

6.c. Adams operations and norm maps. From here on until the end of the paper we suppose that, in addition to the Standing Hypotheses, R also satisfies the following two additional hypotheses:

- (1) The \mathbb{F}_p -algebra $\overline{R} \otimes_{\mathbb{F}_p} \mathbb{F}_p^c$ contains only finitely many orthogonal idempotents.
- (2) For any non-ramified extension L of \mathbb{Q}_p we set $\Delta = \text{Gal}(L/\mathbb{Q}_p)$; then the group $\Lambda(R_L)$ is a \mathbb{Z}_p -module which is Δ -cohomologically trivial and for any p -subgroup Γ of Δ , $H^1(\Gamma, \mathcal{M}(R_L)) = \{1\}$.

For an integer n and a virtual character χ of G , recall that we define $\psi^n \chi$ by the rule that for $g \in G$ we have $\psi^n \chi(g) = \chi(g^n)$. Clearly $\psi^n \chi$ is a central function on G ; in fact one sees easily by Newton's formulas that $\psi^n \chi$ is a virtual character of G . In this subsection we use these Adams operations on characters to define Adams operations on the group $\text{Det}(R[G]^\times)$.

For $\text{Det}(x) \in \text{Det}(R[G]^\times)$ and for an integer n we define $\psi^n \text{Det}(x)$ to be the character function given by the rule that for a virtual character χ of G

$$\psi^n \text{Det}(x)(\chi) = \text{Det}(x)(\psi^n \chi).$$

Then, as per Theorem 1 in [CNT] (see also Sect. 9 in [T]), we have:

Theorem 6.4. *For any integer n , $\psi^n \text{Det}(R[G]^\times) \subset \text{Det}(R[G]^\times)$.*

The proof of this result is in many ways similar to the proof given in [CNT] and Sect. 9 of [T] (when R is the valuation ring of a finite non-ramified extension of \mathbb{Q}_p). The details are provided in an Appendix. However, note that the proof in [CNT] and [T] requires a number of modifications. In particular, one issue that arises at the conclusion of the proof (see pages 114-115 in [T]) is the following: for a finite non-ramified extension L of \mathbb{Q}_p , if we set $R_L = R \otimes \mathcal{O}_L$ then we do not necessarily know that:

$$\mathcal{N}_{L/\mathbb{Q}_p}(\text{Det}(R_L[G]^\times)) = \text{Det}(R[G]^\times);$$

and this is because we do not necessarily know that the group of norms $N_{L/\mathbb{Q}_p}(R_L^\times)$ coincides with R^\times . However, we do have from Theorem 4.3 of [CPT1]:

Theorem 6.5. *Let G be a p -group. With the above notation we have the equality*

$$\mathcal{N}_{L/\mathbb{Q}_p}(\text{Det}(1 + I(R_L[G]))) = \text{Det}(1 + I(R[G])).$$

This is a key-result, as is explained in the Appendix.

The proof of Theorem 6.4 then follows very closely the proof given in [CNT] and [T] by using the above result together with the decomposition

$$\text{Det}(R[G]^\times) = \text{Det}(1 + I(R[G])) \times R^\times;$$

However, the special case where G is a p -group is now considerably more involved. The details are provided in the Appendix.

For future reference note that if H is a subgroup of G , then, by definition, for $\text{Det}(y) \in \text{Det}(R[H]^\times)$ we have,

$$\text{Ind}_H^G(\text{Det}(y))(\chi) = \text{Det}(y)(\text{Res}_H^G(\chi))$$

and so

$$\begin{aligned} (\psi^n \text{Ind}_H^G(\text{Det}(y)))(\chi) &= \text{Ind}_H^G(\text{Det}(y))(\psi^n \chi) \\ &= \text{Det}(y)(\text{Res}_H^G(\psi^n \chi)) = \text{Det}(y)(\psi^n \text{Res}_H^G(\chi)) \\ &= (\psi^n \text{Det}(y))(\text{Res}_H^G(\chi)) = (\text{Ind}_H^G \psi^n \text{Det}(y))(\chi). \end{aligned}$$

Thus we have shown that

Lemma 6.6. *For $H < G$, ψ^n commutes with the map Ind_H^G on determinants.*

6.d. The group logarithm for arbitrary finite groups. We define

$$\Psi : \text{Det}(R[G]^\times) \rightarrow \text{Det}(R[G]^\times)$$

by the rule $\Psi(\text{Det}(x)) = \psi^p \text{Det}(F(x))$ and we define

$$v_G(\text{Det}(x)) = \phi \circ \log \left(\frac{\text{Det}(x)^p}{\Psi(\text{Det}(x))} \right).$$

We begin this subsection by showing the following generalization of Theorem 3.3 in 3.a:

Theorem 6.7. *We have $v_G(\text{Det}(R[G]^\times)) \subset pR[C_G]$.*

In the sequel we shall write v'_G for $p^{-1}v_G$.

Proof. Suppose first that G is a p -group; the result then follows immediately from Theorem 3.3 and 6.b.

Suppose next that G is a \mathbb{Q}_p - p -elementary group. We adopt the notation of §5: We write $G = C \rtimes P$ and we have the decomposition

$$R[G] = \oplus_m R[m] \circ P.$$

We consider the restriction map $\text{res} : R[m] \circ P \rightarrow M_{|A_m|}(R[m][H_m])$ and form the composite

$$R[m] \circ P \rightarrow M_{|A_m|}(R[m][H_m]) \xrightarrow{\text{Tr}} R[m][H_m]^{A_m} \rightarrow (R[m][H_m]/I_{H_m})^{A_m} = R[m][C_{H_m}]^{A_m}$$

where the map $R[m][H_m]^{A_m} \rightarrow (R[m][H_m]/I_{H_m})^{A_m}$ is induced by the natural map $R[m][H_m] \rightarrow R[m][H_m]/I_{H_m}$, where of course A_m acts on both H_m and $R[m]$. Recall that here P acts on $R[m] \circ P$ by conjugation on P and its natural action on $R[m]$. Since A_m is cyclic, we know $[P, P] \subset H_m$, so for any $g, k \in P$ we have $g^k - g$ lies in the $R[G]$ -ideal generated by I_{H_m} ; hence $(\text{Tr} \circ \text{res})(I_P) \subset I_{H_m}$ and we have constructed a map

$$H_0(\text{Tr}) : H_0(P, R[m] \circ P) \rightarrow H_0(H_m, R[m][H_m])^{A_m}.$$

Observe that $R[m][C_{H_m}]^{A_m}$ is spanned over $R[m]^{A_m}$ by the elements $(\text{Tr} \circ \text{res})(\zeta h)$ for $h \in H_m$ and ζ a primitive m -th root of unity; this shows that $H_0(\text{Tr})$ is surjective.

Lemma 6.8. *We have the commutative diagram:*

$$\begin{array}{ccccc} \text{Det}(R[G]^\times) & \xrightarrow{\sim} & \oplus_m \text{Det}(R[m] \circ P^\times) & \xrightarrow{\sim} & \oplus_m \text{Det}(R[m][H_m]^\times)^{A_m} \\ \downarrow v'_G & & \downarrow & & \downarrow \oplus v_m^{A_m} \\ H_0(G, R[G]) & \xrightarrow{\sim} & \oplus_m H_0(G, R[m] \circ P) & \xrightarrow{H_0(\text{Tr})} & \oplus_m H_0(H_m, R[m][H_m])^{A_m} \end{array}$$

and $H_0(\text{Tr})$ is an isomorphism.

Proof. The diagram comes from Theorem 5.1. We have shown $H_0(\text{Tr})$ to be surjective, and the terms in the lower row are R -torsion free with the same rank (as is seen by tensoring by N), and so $H_0(\text{Tr})$ is injective. \square

Lemma 6.9. *We have*

$$\oplus_m H_0(H_m, R[m][H_m])^{A_m} \cong \oplus_m R[m][C_{H_m}]^{A_m} \cong R[C_G].$$

Proof. The result is clear if $C = \{1\}$, so suppose that C is non-trivial and choose a prime number l , that divides $|C|$ and write $|C| = l^r q$ with q coprime to l . Let $c \in C$ denote an element in C of order l and set $\overline{C} = G/C$. We can then write $R[C_G] = R[C_{\overline{C}}] \oplus (1-c)R[C_G]$ where $(1-c)R[C_G]$ denotes $\ker(R[C_G] \rightarrow R[C_{\overline{C}}])$. We then have the decompositions

$$(1-c)R[C] = \oplus_m R[m], \quad (1-c)R[G] = \oplus_m R[m] \circ P$$

where m runs through all the divisors of $|C|$ which are divisible by l^r .

$$\begin{aligned} H_0(G, (1-c)R[G]) &= H_0(G, \oplus_m R[m] \circ P) = \oplus_m H_0(G, R[m] \circ P) \\ &= \oplus_m H_0(P, R[m] \circ P) \stackrel{H_0(Tr)}{\cong} \oplus_m H_0(H_m, R[m][H_m])^{A_m}. \end{aligned}$$

The result then follows by induction on the group order (which gives the result for \overline{G}). \square

To complete the proof of Theorem 6.7 for arbitrary G we recall from Theorem 4.3 that we can write $m \cdot 1_G = \sum_H n_H \text{Ind}_H^G(\theta_H)$ where n_H are integers, the H are \mathbb{Q}_p - p -elementary subgroups of G and the θ_H are \mathbb{Q}_p -characters of H and m is not divisible by p .

Let $\text{Det}(x) \in \text{Det}(R[G]^\times)$. Using Lemma 6.6 we have

$$\begin{aligned} m \cdot v'_G(\text{Det}(x)) &= \sum_H n_H \cdot v'_G(\text{Ind}_H^G((\theta_H)\text{Det}(x))) \\ &= \sum_H n_H \cdot v'_G(\text{Ind}_H^G(\theta_H(\text{Res}_H^G(\text{Det}(x)))))) \\ &= \sum_H n_H \cdot \text{Ind}_H^G(v'_H(\theta_H(\text{Res}_H^G(\text{Det}(x)))))) \end{aligned}$$

and so by the result for \mathbb{Q}_p - p -elementary groups we know that $m \cdot v'_G(\text{Det}(x)) \in pR[C_G]$. The result is then shown since m acts as an automorphism on $R[C_G]$. \square

6.e. Oliver's map $\Theta_{R[G]}$ for arbitrary finite groups. In this section, among other results, we provide a more direct construction of Oliver's isomorphism $\Theta_{R[G]}$ of Theorem 1.7. (Recall that R satisfies the additional hypotheses stated in the beginning of §6.c.)

For a $\mathbb{Z}[G]$ -module M we recall the bar resolution

$$\cdots \rightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G] \otimes_{\mathbb{Z}} M \xrightarrow{\partial_2} \mathbb{Z}[G] \otimes_{\mathbb{Z}} M \xrightarrow{\partial_1} M$$

which computes the homology groups $H_*(G, M)$. (See for instance 13.2 in [HS].) Explicitly for $m \in M$, and $g, h \in G$ we have

$$\begin{aligned} \partial_1(g \otimes m) &= (1 - g)m \\ \partial_2(g \otimes h \otimes m) &= h \otimes m - gh \otimes m + g \otimes hm. \end{aligned}$$

In what follows an element of $H_1(G, M)$ will be represented as an element of the quotient $\ker \partial_1 / \text{Im } \partial_2$ without further reference.

Let G_r again denote the subset of p -regular elements of G ; i.e. the subset of elements of G whose order is coprime to p . We shall be particularly interested in the $R[G]$ -module $R[G_r]$ where G acts on the left on G_r by conjugation; that is to say ${}^g h = ghg^{-1}$ for $g \in G$, $h \in G_r$.

We shall view $R[G_r]$ as a Ψ -module (as in the Introduction) by the rule

$$\Psi(\sum_g s_g g) = \sum_g F(s_g) g^p$$

for $s_g \in R$, $g \in G_r$. Note that the actions of G and Ψ commute, so that we may view $R[G_r]$ as a $G \times \Psi$ -module. We write $H_1(G, R[G_r])^\Psi$ and $H_1(G, R[G_r])_\Psi$ for the groups of Ψ -invariants and Ψ -covariants of $H_1(G, R[G_r])$ in the usual way.

6.e.1. *The map ω_G .* See Theorem 12.9 in [O5]. We also view $R[G]$ as a left G -module by conjugation; so that ${}^g(\sum_h s_h h) = \sum_h s_h \cdot ghg^{-1}$. We may then identify $H_0(G, R[G])$ with $R[C_G]$. Recall that we have previously defined the R -linear map $\phi : R[G] \rightarrow R[C_G]$ by mapping each group element to its conjugacy class. Recall also that each group element $g \in G$ may be written uniquely as $g = g_r g_p$ where g_r has order coprime to p , g_p has p -power order, and g_r and g_p commute.

Proposition 6.10. *The map*

$$\omega_G(\sum_g s_g g) = \sum_g g \otimes s_g g_r$$

induces a homomorphism

$$\omega_G : H_0(G, R[G]) \rightarrow H_1(G, R[G_r]).$$

Proof. For simplicity, write ω instead of ω_G . Note that $\partial_1(\omega(g)) = \partial_1(g \otimes g_r) = g_r - gg_r g^{-1} = 0$, since g and g_r commute. To see that we get a well-defined homomorphism (that is to say which is independent of choices), we note that

$$\omega({}^g h) = {}^g h \otimes {}^g h_r = {}^g \omega(h)$$

and ${}^g \omega(h)$ and $\omega(h)$ are homologous since the functorial action of G on homology groups is trivial (see for instance Ch. III Proposition 8.3 in [B]). \square

Remark 6.11. For future reference note that if G is p -group, then $G_r = \{1\}$, $H_1(G, R[G_r]) = G^{\text{ab}} \otimes R$, and ω_G is the R -linear map induced by $G \rightarrow G^{\text{ab}} \otimes 1$.

6.e.2. *The map ξ_G .*

Proposition 6.12. *For $u = \sum_{g \in G} s_g g \in \text{GL}_n(R[G])$, with $s_g \in M_n(R)$, we write $u^{-1} = \sum_{h \in G} t_h h$ and set*

$$\xi'_G(u) = \sum_{g,h} g \otimes \text{Tr}(t_h s_g)(hg)_r \in R[G_r].$$

This induces a homomorphism

$$\xi_G : K_1(R[G]) \rightarrow H_1(G, R[G_r])$$

and the element

$$\xi_G(u) = \sum_{g,h} (g - 1) \otimes \text{Tr}(t_h s_g)(hg)_r \in R[G_r]$$

lies in the same homology class as $\xi'_G(u)$ in $H_1(G, R[G_r])$.

Proof. (See Theorem 12.9.i in [O5].) We first show that $\xi' = \xi'_G$ is a homomorphism from $\text{GL}_n(R[G])$ into $H_1(G, R[G_r])$. To this end we first note that

$$1 = \sum_{g,h} t_h s_g h g = \sum_{g,h} s_g t_h g h$$

and so

$$n = \sum_{g,h} \text{Tr}(t_h s_g) h g = \sum_{g,h} \text{Tr}(s_g t_h) g h$$

since for given $x \in G$, $\sum_{hg=x} \text{Tr}(t_h s_g) = 0$ unless $x = 1$, in which case of course we have $h = g^{-1}$. We then see that

$$\partial_1 \circ \xi'(u) = \sum_{g,h} \text{Tr}(t_h s_g) (hg)_r - \text{Tr}(t_h s_g) (gh g g^{-1})_r = 0.$$

To see that ξ' is a homomorphism, consider $v = \sum_{a \in G} p_a a \in \text{GL}_n(R[G])$ and write $v^{-1} = \sum_{b \in G} q_b b$. We then see that

$$\xi'(uv) = \sum g a \otimes \text{Tr}(q_b t_h s_g p_a) (bhga)_r.$$

Using the fact that $gh \otimes x - h \otimes x - g \otimes h x h^{-1}$ lies in $\text{Im } \partial_2$, we get the congruence modulo $\text{Im } \partial_2$

$$\begin{aligned} \xi'(uv) &\equiv \sum_{a,b} a \otimes \sum_{h,g} \text{Tr}(q_b t_h s_g p_a) (bhga)_r + \\ &\quad \sum_{h,g} g \otimes \sum_{a,b} \text{Tr}(q_b t_h s_g p_a) (abhg)_r \end{aligned}$$

and using the equalities

$$\begin{aligned} \sum_{h,g} \text{Tr}(q_b t_h s_g p_a) (bhga)_r &= \text{Tr}(q_b p_a) (ba)_r \\ \sum_{a,b} \text{Tr}(p_a q_b t_h s_g) (abhg)_r &= \text{Tr}(s_h r_g) (hg)_r \end{aligned}$$

we find $\xi'(uv) = \xi'(u) + \xi'(v)$ in $H_1(G, R[G_r])$. Next observe that, since ξ' is homomorphism into an abelian group, it factors through $K_1(R[G])$.

To conclude we note that setting $h = 1$ in $gh \otimes x = h \otimes x + g \otimes h x h^{-1}$ we get that $\xi(u) - \xi'(u)$ is in $\text{Im } \partial_2$. \square

The following result shows how the above map ξ_G relates to the p -group logarithmic differential map in 3.3 of [CPT1].

Lemma 6.13. *Suppose that G is a p -group, so that $G_r = \{1\}$. Then, under the identification*

$$\Omega_{R[G]/R}^1 = G^{\text{ab}} \otimes_{\mathbb{Z}} R = \frac{I_G}{I_G^2} \otimes_{\mathbb{Z}} R = H_1(G, R),$$

we have the equality $\xi_G(u) = d \log(u)$ for $u \in R[G]^\times$. Let Ψ denote the F -semi-linear map on $\Omega_{R[G]/R}^1$ (so that for $x \in I_G/I_G^2$, $r \in R$, we have $\Psi(r dx) = F(r) \Psi(dx)$) and which has the property that $p \Psi \circ d = d \circ \Psi$ (see Proposition 3.18 in [CPT1]). Then we have the equality for $u \in R[G]^\times$

$$(1 - \Psi) \xi_G(u) = d(p^{-1} \mathcal{L}_G(u)) = d(v'_G(\text{Det}(u))) = \omega_G \circ v'_G(\text{Det}(u)).$$

Proof. Let $u = \sum_g s_g g$, $u^{-1} = \sum_h t_h h$. Since $(g-1)h \equiv (g-1) \pmod{I_G^2}$, we obtain the congruence $\pmod{I_G^2}$:

$$du \cdot u^{-1} = \sum_{g,h} (g-1)h \otimes s_g t_h \equiv \sum_{g,h} (g-1) \otimes s_g t_h \equiv \xi_G(u)$$

and also

$$\begin{aligned} (1 - \Psi)\xi_G(u) &= (1 - \Psi)d \log(u) = \\ &= d((1 - \Psi/p) \log(u)) = d(p^{-1} \mathcal{L}_G(u)) = d(v'_G(\text{Det}(u))). \end{aligned}$$

□

For this short paragraph G is still a p -group so that $G_r = \{1\}$. As in the final part of 6.b we can decompose the map $v'_G = v_\varepsilon \times v_I$ via the decomposition $\text{Det}(R[G]^\times) = R^\times \times \text{Det}(1 + I_G)$ and note that v_ε coincides with the map $p^{-1} \mathcal{L}_R$ of 6.b. Using the above and noting that $H_1(G, R) = R \otimes G^{\text{ab}}$ we see that the exact sequence

$$\text{Det}(1 + I_G) \xrightarrow{v'_G} \phi(I_G) \xrightarrow{\omega_G} \frac{R}{(1 - F)R} \otimes G^{\text{ab}} \rightarrow 0$$

of (3.3) affords the further exact sequence

$$\text{Det}(1 + I_G) \xrightarrow{v'_G \times \xi_G} \phi(I_G) \oplus H_1(G, R) \xrightarrow{\omega_G + (\Psi - 1)} H_1(G, R) \rightarrow 0.$$

By adding this to the exact sequence

$$R^\times \xrightarrow{p^{-1} \mathcal{L}_R \oplus \theta} R \oplus \Lambda(R) \xrightarrow{T} R \rightarrow 0$$

of (6.3) we get the further exact sequence

$$(6.5) \quad \text{Det}(R[G]^\times) \xrightarrow{s_G} R[C_G] \oplus H_1(G, R) \oplus \Lambda(R) \xrightarrow{T_G} H_1(G, R) \oplus R \rightarrow 0$$

where $s_G(\text{Det}(z)) = (v'_G \times \xi_G \times \theta_G)$ and for $(x, y, z) = (x_\varepsilon \oplus x_I, y_I, z_\varepsilon)$ we have $T_G(x, y, z) = (\omega_G(x_I) + (\Psi - 1)y, x_\varepsilon - v'_\varepsilon(z))$.

Lemma 6.14. *For an arbitrary finite group G let $J = J_{R[G]}$ denote the Jacobson radical of $R[G]$. Then $\text{SK}_1(R[G])$ is contained in $K'_1(R[G], J)$. If G is abelian, then $\text{SK}_1(R[G]) = \{1\}$.*

Proof. Consider the Wedderburn decomposition

$$\mathbb{Z}_p[G]/J = \prod_i M_{n_i}(k_i)$$

where each k_i is a finite field of characteristic p and let W_i denote the ring of Witt vectors of k_i . By the Standing Hypotheses we know that $\text{SK}_1(R \otimes_{\mathbb{Z}_p} W_i) = \{1\}$. We

write $\overline{R} = R/pR$ which is a domain. Observe that we have the commutative diagram

$$\begin{array}{ccccccc}
 & & & & K_1(R \otimes_{\mathbb{Z}_p} W_i, (p)) & \rightarrow & 1 + pR \otimes_{\mathbb{Z}_p} W_i \\
 & & & & \downarrow & & \downarrow \\
 1 & \rightarrow & SK_1(R \otimes_{\mathbb{Z}_p} W_i) & \rightarrow & K_1(R \otimes_{\mathbb{Z}_p} W_i) & \xrightarrow{\det} & (R \otimes_{\mathbb{Z}_p} W_i)^\times \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \rightarrow & SK_1(\overline{R} \otimes_{\mathbb{Z}_p} W_i) & \rightarrow & K_1(\overline{R} \otimes_{\mathbb{Z}_p} W_i) & \xrightarrow{\det} & (\overline{R} \otimes_{\mathbb{Z}_p} W_i)^\times \rightarrow 1 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 1 & & 1
 \end{array}$$

Since $1 + pR \otimes_{\mathbb{Z}_p} W_i$ injects into $(R \otimes_{\mathbb{Z}_p} W_i)^\times$ and since $K_1(R \otimes_{\mathbb{Z}_p} W_i)$ surjects onto $K_1(\overline{R} \otimes_{\mathbb{Z}_p} W_i)$, because p is in the Jacobson radical of $R \otimes_{\mathbb{Z}_p} W_i$, we see that $SK_1(R \otimes_{\mathbb{Z}_p} W_i) \rightarrow SK_1(\overline{R} \otimes_{\mathbb{Z}_p} W_i)$ maps onto and so $SK_1(\overline{R} \otimes_{\mathbb{Z}_p} W_i) = \{1\}$. This then shows $SK_1(R[G])$ is contained in $\text{Im}(\text{GL}(R[G], J))$, as required.

Now suppose that G is abelian. By Lemma 2.2 (b) we know that if we have $x \in \text{GL}(R[G], J)$ with $\text{Det}(x) = 1$, then x is elementary. The result then follows since by the above $SK_1(R[G]) \subset \text{Im}(\text{GL}(R[G], J))$. \square

Lemma 6.15. *For an arbitrary finite group G , ξ_G is trivial on $SK_1(R[G])$. We shall therefore henceforth feel free to view ξ_G as being defined on $\text{Det}(R[G]^\times)$.*

Proof. We know from Theorem 4.6 that $SK_1(R[G])$ is computable by induction from \mathbb{Q}_p - p -elementary groups. Thus, by Theorem 5.5, it will suffice to prove the result for p -groups. However, from Lemma 6.13 we know that $\xi(SK_1(R[G])) = \xi(SK_1(R[G^{\text{ab}}]))$ and Lemma 6.14 above shows that $SK_1(R[G^{\text{ab}}]) = 1$. \square

Lemma 6.16. *Suppose we have a surjection of finite groups $\alpha : \tilde{G} \rightarrow G$. Then $SK_1(R[G]) \subset \alpha_*(K_1(R[\tilde{G}]))$.*

Proof. By Lemma 6.14 we have $SK_1(R[G]) \subset K'_1(R[G], J_{R[G]})$ where $J_{R[G]}$ denotes the Jacobson radical of $R[G]$. We claim next that $\alpha(J_{R[\tilde{G}]}) = J_{R[G]}$. We have $J_{R[G]} = (J_R, J_{\mathbb{Z}_p[G]})$ and similarly for $J_{R[\tilde{G}]}$; thus it will suffice to show that $\alpha(J_{\mathbb{Z}_p[\tilde{G}]}) = J_{\mathbb{Z}_p[G]}$ and therefore it will suffice to show $\alpha(J_{\mathbb{F}_p[\tilde{G}]}) = J_{\mathbb{F}_p[G]}$. We know that the quotient rings $\mathbb{F}_p[\tilde{G}]/J_{\mathbb{F}_p[\tilde{G}]}$ and $\mathbb{F}_p[G]/J_{\mathbb{F}_p[G]}$ are semisimple with the former mapping onto the latter with kernel a two-sided ideal. Thus we have a decomposition

$$\mathbb{F}_p[\tilde{G}]/J_{\mathbb{F}_p[\tilde{G}]} = X \oplus (\mathbb{F}_p[G]/J_{\mathbb{F}_p[G]})$$

which by lifting of idempotents lifts to a splitting

$$\mathbb{F}_p[\tilde{G}] = X' \oplus Y$$

where $Y/J_Y \cong \mathbb{F}_p[G]/J_{\mathbb{F}_p[G]}$; and so we see that indeed $J_{\mathbb{F}_p[\tilde{G}]}$ maps onto $J_{\mathbb{F}_p[G]}$. Finally we conclude since

$$\text{Mat}(R[\tilde{G}], J_{R[\tilde{G}]}) = \text{GL}(R[\tilde{G}], J_{R[\tilde{G}]}), \quad \text{Mat}(R[G], J_{R[G]}) = \text{GL}(R[G], J_{R[G]})$$

and by the above $\text{Mat}(R[\tilde{G}], J_{R[\tilde{G}]})$ maps onto $\text{Mat}(R[G], J_{R[G]})$. \square

6.e.3. *The map θ_G .* First we need:

Lemma 6.17. *Let $\mathcal{E}_p(G)$ denote the set of \mathbb{Q}_p -*p*-elementary subgroups of G . Then*

$$(a) \varinjlim_{H \in \mathcal{E}_p(G)} H_0(H, R[H]) = H_0(G, R[G]);$$

$$(b) \varinjlim_{H \in \mathcal{E}_p(G)} H_1(H, R[H_r]) = H_1(G, R[G_r]).$$

(c) *Suppose that for each subgroup H of G , we are given a subgroup $M(H)$ of $\text{Det}(R[H]^\times)$ such that the $M(H)$ are natural with respect to inclusion, so that we have $\text{Ind}_H^G(M(H)) \subset M(G)$. Suppose further that for each H the quotient group $\text{Det}(R[H]^\times)/M(H)$ is a \mathbb{Z}_p -module and that $\text{Det}(R[G]^\times)/M(G)$ is torsion free; then the natural map*

$$\varinjlim_{H \in \mathcal{E}_p(G)} \text{Det}(R[H]^\times)/M(H) \rightarrow \text{Det}(R[G]^\times)/M(G)$$

is surjective.

Proof. Let $d \in \text{Det}(R[G]^\times)$. To see that (c) holds observe that by 4.b for some m coprime to p we can write

$$m \cdot d = \sum_{H \in \mathcal{E}_p(G)} n_H \text{Ind}_H^G(\theta_H)d = \sum_{H \in \mathcal{E}_p(G)} n_H \text{Ind}_H^G(\theta_H \text{Res}_H^G(d))$$

which belongs to $\sum_{H \in \mathcal{E}_p(G)} \text{Ind}_H^G(\text{Det}(R[H]^\times))$. The result then follows since all the $\text{Det}(R[H]^\times)/M(H)$ are \mathbb{Z}_p -modules where m acts as an automorphism.

For (a) and (b) by Theorem 4.4 it suffices to show that the right-hand terms are generated over $\mathcal{E}_p(G)$. This holds for (a) because $\mathcal{E}_p(G)$ contains all cyclic subgroups of G . To see that (b) holds, note that for $g \in G_r$ we have

$$H_1(G, \sum_{h \in G} Rg^h) = H_1(Z_G(g), Rg) = H_1(Z_G(g)_p, Rg)$$

where $Z_G(g)_p$ denotes a p -Sylow subgroup of the centralizer $Z_G(g)$. The result then follows since $Z_G(g)_p \cdot \langle g \rangle \in \mathcal{E}_p(G)$. \square

Lemma 6.18. *For any finite group G we have that $\text{Det}(R[G]^\times)/\ker(v'_G \times \xi_G)$ is a \mathbb{Z}_p -module.*

Proof. Since $\text{Im}(\xi_G)$ is a finite abelian p -group, it will suffice to show that the quotient $\text{Det}(R[G]^\times)/\ker(v'_G)$ is a \mathbb{Z}_p -module. This follows for p -groups by (6.3) and (3.3). The result for \mathbb{Q}_p -*p*-elementary groups then follows from Theorem 5.1. The result for general finite groups follows from (c) above using Theorem 6.7 to see that $\text{Det}(R[G]^\times)/\ker(v'_G)$ is torsion free. \square

We now define a group $\Lambda(R[G])$ and maps θ_G and $\tilde{\mathcal{L}}_{R[G]}$; these appear in the statement of Proposition 6.20.

Let $d : G_0(\mathbb{Q}_p[G]) \rightarrow G_0(\mathbb{F}_p[G])$ denote the decomposition map, which is a surjective λ -ring homomorphism which admits a natural splitting. Thus the isomorphism classes of $\mathbb{F}_p[G]$ -modules are identified with Brauer modular characters in characteristic zero; so, in the sequel, for $\xi \in G_0(\mathbb{Q}_p[G])$ we write $d(\xi)$ both for the image of ξ in $G_0(\mathbb{F}_p[G])$ and for its Brauer lift in $\xi \in G_0(\mathbb{Q}_p[G])$. Note also that for $\theta \in G_0(\mathbb{Q}_p[G])$

we have the equality $d(\xi) \cdot \theta = d(\xi) \cdot (d(\theta) + (\theta - d(\theta))) = d(\xi \cdot \theta)$. We define $\mathcal{M}(R[G])$ as

$$\{\text{Det}(x) \in \text{Det}(R[G]^\times) \mid \Psi(\text{Det}(x))(d(\xi)) = \text{Det}(x)(d(\xi))^p \text{ for all } \xi \in G_0(\mathbb{Q}_p[G])\}.$$

We assert that the $G_0(\mathbb{Q}_p[G])$ -Green structure of $\text{Det}(R[G]^\times)$ makes $\mathcal{M}(R[G])$ a $G_0(\mathbb{Q}_p[G])$ -Green submodule: indeed, for $\theta \in G_0(\mathbb{Q}_p[G])$ recall we have $\theta \cdot \text{Det}(x)(\xi) = \text{Det}(x)(\xi\bar{\theta})$, and so for $\text{Det}(x) \in \mathcal{M}(R[G])$ we have

$$\begin{aligned} \Psi(\theta \cdot \text{Det}(x))(d(\xi)) &= F((\theta \cdot \text{Det}(x)))(\psi^p d(\xi)) = F((\theta \cdot \text{Det}(x)))(d(\psi^p \xi)) \\ &= F(\text{Det}(x))(d(\psi^p \xi) \cdot F(\bar{\theta})) = F(\text{Det}(x))(d(\psi^p \xi) \cdot dF(\bar{\theta})) \\ &= F(\text{Det}(x))(d(\psi^p \xi) \cdot \psi^p d(\bar{\theta})) = F(\text{Det}(x))(\psi^p d(\xi \cdot \bar{\theta})) \\ &= \Psi(\text{Det}(x))(d(\xi\bar{\theta})) = \text{Det}(x)(d(\xi\bar{\theta}))^p = (\theta \cdot \text{Det}(x)(d(\xi)))^p. \end{aligned}$$

For future reference we note that $\mathcal{M}(R[G]) \supset \ker d \cdot \text{Det}(R[G]^\times)$.

We define:

$$\begin{aligned} \text{Det}'(R[G]^\times) &= \text{Det}'(R[G]^\times) \otimes_{G_0(\mathbb{Q}_p[G])} G_0(\mathbb{F}_p[G]) = \frac{\text{Det}(R[G]^\times)}{\ker d \cdot \text{Det}(R[G]^\times)} \\ \mathcal{M}'(R[G]) &= \frac{\mathcal{M}(R[G])}{\ker d \cdot \text{Det}(R[G]^\times)} \end{aligned}$$

and we set:

$$\Lambda(R[G]) = \text{Det}'(R[G]^\times) / \mathcal{M}'(R[G]) = \text{Det}(R[G]^\times) / \mathcal{M}(R[G]).$$

The map θ_G is defined to be the composite

$$\theta_G : \text{Det}(R[G]^\times) \rightarrow \text{Det}'(R[G]^\times) \rightarrow \Lambda(R[G]).$$

We now define

$$\tilde{\mathcal{L}}_{R[G]} : \Lambda(R[G]) \rightarrow \prod_i R \otimes_{\mathbb{Z}_p} W_i = H_0(G, R[G_r])$$

where the product extends over the irreducible modular characters of $G_0(\mathbb{F}_p[G])$, and where $\tilde{\mathcal{L}}_{R[G]}$ is the product of the logarithmic maps

$$\frac{1}{p} \mathcal{L}_{R \otimes_{\mathbb{Z}_p} W_i} : (R \otimes_{\mathbb{Z}_p} W_i)^\times \rightarrow R \otimes_{\mathbb{Z}_p} W_i$$

followed by identification of the free R -module on the p -regular conjugacy classes of G , namely $H_0(G, R[G_r])$, with $\prod_i R \otimes_{\mathbb{Z}_p} W_i$, by evaluation on the irreducible modular characters χ_i of G (see Brauer's Theorem in 18.2 of [S] and in particular Exercise 4 in 18.2). To be more precise, writing χ_i for the irreducible character associated to the i -th component of the above description, explicitly for each $x \in R[G]^\times$ we may view $\tilde{\mathcal{L}}_{R[G]} \circ \theta_G(\text{Det}(x))$ as being given by the value

$$\prod_i \frac{1}{p} \mathcal{L}_{R \otimes_{\mathbb{Z}_p} W_i}(\text{Det}_{\chi_i}(x)).$$

We view $\text{Det}'(R[G]^\times)$ as a Green module for the ring $G_0(\mathbb{F}_p[G])$, and by the above $\mathcal{M}'(R[G])$ is a sub-Green module of $\text{Det}'(R[G]^\times)$, and so $\Lambda(R[G])$ is a quotient Green module for $G_0(\mathbb{F}_p[G])$.

For each $H \subset G$, in the usual way we have the induction map $\text{Ind}_H^G : \text{Det}'(R[H]^\times) \rightarrow \text{Det}'(R[G]^\times)$ and hence we have a natural map $\Lambda(R[H]) \rightarrow \Lambda(R[G])$.

Proposition 6.19. *For each finite group G , the group $\Lambda(R[G])$ is a pro- p -group and*

$$\varinjlim_{H \in \mathcal{E}_p(G)} \Lambda(R[H]) = \Lambda(R[G]).$$

Proof. The proof proceeds in three steps.

First Step: Note that if $p \nmid |G|$, then we have a decomposition

$$R[G] \cong \prod_i M_{n_i}(R \otimes_{\mathbb{Z}_p} W_i)$$

for rings of integers W_i of non-ramified extensions of \mathbb{Q}_p . The result in this case therefore follows from Hypothesis (2) in 6.c.

Second step: Suppose now that G is \mathbb{Q}_p - l -elementary for some prime number $l \neq p$. We may write $G = (C' \times C_{p^m}) \rtimes L$ where L is an l -group, C_{p^m} is a cyclic group of order p^m for some $m \geq 0$ and C' is a cyclic group with order prime to lp . If $m = 0$, then we are in the situation dealt with in the first step and so we now suppose that $m > 0$. We set $G' = C' \rtimes L$, and note that the natural quotient $G \rightarrow G'$ is split.

Every irreducible character of G can be written in the form $\text{Ind}_H^G(\chi)$ for χ an abelian character of a subgroup H of G which contains $C' \times C_{p^m}$; moreover, the irreducible characters of G which are inflated from G' arise precisely from those χ which are trivial on C_{p^m} .

Recall that $\langle c \rangle = C_{p^m}$. We have the split exact sequence:

$$1 \rightarrow \text{Det}(1 + (1 - c)R[G]) \rightarrow \text{Det}(R[G]^\times) \hookrightarrow \text{Det}(R[G']^\times) \rightarrow 1$$

and the surjection $G \rightarrow G'$ induces an isomorphism $G_0(\mathbb{F}_p[G]) \cong G_0(\mathbb{F}_p[G'])$. Thus, from the definition of $\mathcal{M}(R[G])$, we have the direct decomposition

$$\mathcal{M}(R[G]) = \text{Det}(1 + (1 - c)R[G]) \times \mathcal{M}(R[G'])$$

and hence the natural map $\Lambda(R[G]) \rightarrow \Lambda(R[G'])$ is an isomorphism, and we are now done by Step 1.

Third Step: To prove the final part of the Proposition, we use Brauer's induction theorem for modular characters (see Theorem 39 in [S]). By this result we know that for each prime $l \neq p$, we can find a positive integer d_l , prime to l such that for each $\lambda \in \Lambda(R[G])$ we know that $d_l \lambda$ is in the image of induction from \mathbb{Q}_p - l -elementary subgroups of G . So varying over the primes different l from p , by the above we know that we can find a non-negative integer m such that $p^m \Lambda(R[G])$ is a finite sum of pro- p -groups.

To conclude we use Brauer's induction theorem for \mathbb{Q}_p - p -elementary groups expressed in terms of modular characters (by applying the decomposition map to the standard form of Brauer's theorem) together with the Green module structure of $\Lambda(R[G])$ over $G_0(\mathbb{F}_p[G])$; this then shows that $\Lambda(R[G])$ is generated by induction

from \mathbb{Q}_p - p -elementary groups, and so by Theorem 4.4 we deduce

$$\varinjlim_{H \in \mathcal{E}_p(G)} \Lambda(R[H]) = \Lambda(R[G]).$$

□

As previously we set $M(H) = \ker(v'_H \times \xi_H)$. We conclude this subsection by noting that the maps

$$\begin{aligned} v'_H &: \text{Det}(R[H]^\times)/M(H) \rightarrow H_0(H, R[H]), \\ \xi_H &: \text{Det}(R[H]^\times)/M(H) \rightarrow H_1(H, R[H]), \end{aligned}$$

afford commutative diagrams

$$\begin{array}{ccc} \varinjlim_{H \in \mathcal{E}_p(G)} \text{Det}(R[H]^\times)/M(H) & \rightarrow & \varinjlim_{H \in \mathcal{E}_p(G)} H_0(H, R[H]) \\ \downarrow & \searrow \widehat{v}'_G & \downarrow \\ \text{Det}(R[G]^\times)/M(G) & \rightarrow & H_0(G, R[G]) \end{array}$$

and

$$\begin{array}{ccc} \varinjlim_{H \in \mathcal{E}_p(G)} \text{Det}(R[H]^\times)/M(H) & \rightarrow & \varinjlim_{H \in \mathcal{E}_p(G)} H_1(H, R[H]) \\ \downarrow & \searrow \widehat{\xi}_G & \downarrow \\ \text{Det}(R[G]^\times)/M(G) & \rightarrow & H_1(G, R[G]) \end{array}$$

where we denote the diagonal maps by \widehat{v}'_G and $\widehat{\xi}_G$ respectively.

6.e.4. *An exact sequence.* Recall that we identify $H_0(G, R[G])$ and $R[C_G]$. We define $s_G = v'_G \times \xi_G \times \theta_G$ and we let $\rho_G : H_0(G, R[G]) \rightarrow H_0(G, R[G_r])$ be the R -linear map induced by mapping each group element $g \in G$ to its p -regular component g_r .

Proposition 6.20. *The following sequence is exact:*

$$(6.6) \quad \text{Det}(R[G]^\times) \xrightarrow{s_G} H_0(G, R[G]) \oplus H_1(G, R[G_r]) \oplus \Lambda(R[G]) \rightarrow \\ \xrightarrow{T_G} H_1(G, R[G_r]) \oplus H_0(G, R[G_r]) \rightarrow 0$$

where $T_G(x \oplus y \oplus z) = (\omega_G(x) - (1 - \Psi)y) \oplus (\rho_G(x) - \widetilde{\mathcal{L}}_{R[G]}(z))$.

Proof. Since formation of direct limits over $\mathcal{E}_p(G)$ (which need not necessarily form a directed system) is a right exact functor, by Lemma 6.17 we are reduced to showing that (6.6) is exact when G is \mathbb{Q}_p - p -elementary.

We again use the notation of Section 5 and write $G = C \rtimes P$. First note that, as H_m is a p -group, clearly $H_{m,r} = \{1\}$. For ease of notation, we will write $H = H_m$, $A = A_m$ but still retain $R[m]$ so that we are reminded of the corresponding extension of scalars. Next observe that as in (6.5) we have the exact sequence:

$$\begin{aligned} \text{Det}(R[m][H]^\times) &\xrightarrow{s_H} H_0(H, R[m][H]) \oplus H_1(H, R[m]) \oplus \Lambda(R[m]) \\ &\xrightarrow{T_H} H_1(H, R[m]) \oplus H_0(H, R[m]) \rightarrow 0 \end{aligned}$$

which we rewrite as

$$(6.7) \quad \begin{aligned} \text{Det}(R[m][H]^\times) &\xrightarrow{s_H} H_0(H, R[m][H]) \oplus (H^{\text{ab}} \otimes R[m]) \oplus \Lambda(R[m]) \\ &\xrightarrow{T_H} (H^{\text{ab}} \otimes R[m]) \oplus R[m] \rightarrow 0. \end{aligned}$$

We then take A -fixed points to obtain the sequence:

$$(6.8) \quad \begin{aligned} \text{Det}(R[m][H]^\times)^A &\xrightarrow{s_H^A} H_0(H, R_m[H])^A \oplus (H^{\text{ab}} \otimes R[m])^A \oplus \Lambda(R[m])^A \\ &\xrightarrow{T_H^A} (H^{\text{ab}} \otimes R[m])^A \oplus R[m]^A \rightarrow 0 \end{aligned}$$

and we now wish to show that this sequence is exact. For this we first appeal to the fact that the two following sequences are left exact:

$$\begin{aligned} 1 \rightarrow \ker(s_H)^A &\rightarrow \text{Det}(R[m][H]^\times)^A \rightarrow \text{Im}(s_H)^A, \\ 1 \rightarrow \text{Im}(s_H)^A &\rightarrow H_0(H, R[m][H])^A \oplus (H^{\text{ab}} \otimes R[m])^{A_m} \oplus \Lambda(R[m])^A \\ &\xrightarrow{T^A} (H^{\text{ab}} \otimes R[m])^A \oplus R[m]^A. \end{aligned}$$

Considering the terms in the lower sequence (before we take A -fixed points) we see that, because $R[m]$ is $A = A_m$ -free, $H_0(H, R[m][H])$, $H_1(H_m, R[m])$ and (by hypothesis) the $\Lambda(R[m])$ are all A -cohomologically trivial modules, and so $\text{Im}(s_H)$ is also A -cohomologically trivial. Hence the map from $H^1(A, \ker(s_H))$ to $H^1(A, \text{Det}(R[m][H]^\times))$ is an isomorphism; therefore (6.8) is indeed seen to be exact.

We claim that $H_1(H, R[m])^A = H_1(G, R[m])$. By restriction and Shapiro's lemma (using the fact that $R[m]$ is A -free)

$$H_1(G, R[m]) \cong H_1(P, R[m]) \cong H_1(H, R[m]^A)$$

and

$$H_1(H, R[m])^A = (H^{\text{ab}} \otimes R[m])^A = (H^{\text{ab}} \otimes R[m]^A) = H_1(H, R[m]^A).$$

We then sum over m to obtain

$$\begin{aligned} \oplus_m H_1(H_m, R[m])^{A_m} &\cong \oplus_m H_1(G, R[m]) \cong \\ &\cong H_1(G, \oplus_m R[m]) \cong H_1(G, R[C]) \cong H_1(G, R[G_r]). \end{aligned}$$

By Theorem 5.1 we know that

$$\oplus_m \text{Det}(R[m][H_m]^\times)^{A_m} = \oplus_m \text{Det}((R[m] \circ P)^\times) = \text{Det}(R[G]^\times)$$

and by Lemma 6.9 we know that

$$\oplus_m H_0(H_m, R[m][H_m])^{A_m} = \oplus_m R[m][C_{H_m}]^{A_m} = R[C_G]$$

and finally we must show the equality

$$\oplus_m \Lambda(R[H_m])^{A_m} = \Lambda(R[G]).$$

Let N again denote the field of fractions of R and we put

$$\ker(d_m) = \ker(d_G) \cap G_0(N[m] \circ P) \quad \text{and} \quad \text{Im}(d_m) = \text{Im}(d_G) \cap G_0(N[m] \circ P)$$

so that

$$\ker d_G = \oplus_m \ker d_m \quad \text{and} \quad d_G = \oplus_m \text{Im} d_m,$$

and, as usual, ε_m denotes the augmentation map of $N[m][H_m]$. Then, for each divisor m of $|C|$, we fix an abelian character χ of order m and write $G_m = C \rtimes H_m$. Then we see that $\ker d_m$ is the group generated by virtual characters of degree zero of the

form $\text{Ind}_{G_m}^G \chi \cdot (\rho - \rho(1))$ for characters ρ of H_m , and $\text{Im}d_m = \mathbb{Z} \cdot \text{Ind}_{G_m}^G \chi \cdot \epsilon_m$ where ϵ_m is the trivial character of H_m .

We let $I_{P,m}$ denote the 2-sided $R[m] \circ P$ -ideal generated by the augmentation ideal $I(R[m][H_m])$. From the proof of Corollary 5.4 we observe that we have the exact sequence

$$1 \rightarrow \text{Det}(1 + I_{P,m}) \rightarrow \text{Det}(R[m] \circ P) \rightarrow R[m]^{A_m \times} \rightarrow 1$$

where the right-hand map is evaluation on $\text{Ind}_{G_m}^G \chi \cdot \epsilon_m$; we therefore see that

$$(6.9) \quad \begin{aligned} \ker d_G \cdot \text{Det}(R[G]^\times) &\cong \oplus_m \text{Det}(R[m] \circ P^\times)|_{\ker d_m} = \\ &= \oplus_m \text{Det}(1 + I_{P,m})|_{\ker d_m} \cong \oplus_m \text{Det}(1 + I_{P,m}). \end{aligned}$$

Hence we have the commutative diagram with exact rows

$$\begin{array}{ccccc} \ker d_G \cdot \text{Det}(R[G]^\times) & \hookrightarrow & \text{Det}(R[G]^\times) & \twoheadrightarrow & \text{Det}'(R[G]^\times) \\ \downarrow & & \downarrow r & & \downarrow \\ \oplus_m \text{Det}(1 + I(R[m][H_m]))^{A_m} & \hookrightarrow & \oplus_m \text{Det}(R[m][H_m]^\times)^{A_m} & \twoheadrightarrow & \oplus_m \text{Det}'(R[m][H_m]^\times)^{A_m}. \end{array}$$

Indeed, the top sequence is exact by definition; the lower sequence is exact by applying the A_m -fixed point functor to the exact sequence for the definition of $\text{Det}'(R[m][H_m]^\times)$ and then appealing to Proposition 5.3 which shows that $\text{Det}(1 + I(R[m][H_m]))$ is A_m -cohomologically trivial. The central vertical map is the isomorphism $r = \oplus_m r_m$ of Theorem 5.1; and the left-hand vertical arrow is an isomorphism by the above discussion and Theorem 5.1 which shows that $r_m(\text{Det}(1 + I_{P,m})) = \text{Det}(1 + I(R[m][H_m]))^{A_m}$; therefore the right-hand vertical is also an isomorphism.

The second pair of exact sequences that we need are

$$\begin{array}{ccccc} \ker d_G \cdot \text{Det}(R[G]^\times) & \hookrightarrow & \mathcal{M}(R[G]) & \twoheadrightarrow & \mathcal{M}'(R[G]) \\ \downarrow & & \downarrow & & \downarrow \\ \oplus_m \text{Det}(1 + I(R[m][H_m]))^{A_m} & \hookrightarrow & \oplus_m \mathcal{M}(R[m][H_m])^{A_m} & \twoheadrightarrow & \oplus_m \mathcal{M}'(R[m][H_m])^{A_m}. \end{array}$$

Here again the top row is exact by definition; for the lower exact sequence we note that $\ker d_{H_m} \text{Det}(R[m][H_m]^\times) = \text{Det}(1 + I(R[m][H_m]))$, as in (6.9) above, and so we have the exact sequence

$$1 \rightarrow \text{Det}(1 + I(R[m][H_m])) \rightarrow \mathcal{M}(R[m][H_m]) \rightarrow \mathcal{M}'(R[m][H_m]).$$

We again we take A_m -fixed points. The central vertical map is then an isomorphism by the definition of \mathcal{M} and the fact that r is an isomorphism on $\text{Det}(R[G]^\times)$.

Comparing the two pairs of exact sequences and using the hypothesis (2) in 6.c we conclude that

$$\Lambda(R[G]) = \frac{\text{Det}'(R[G])}{\mathcal{M}'(R[G])} = \oplus_m \frac{\text{Det}'(R[m][H_m]^\times)^{A_m}}{\mathcal{M}'(R[m][H_m]^\times)^{A_m}} = \oplus_m \Lambda(R[m][H_m])^{A_m}$$

as required. \square

6.e.5. *The maps δ_α and ι .* Compare to page 288 of [O5]. Given an extension of finite groups

$$(6.10) \quad 1 \rightarrow K \rightarrow \tilde{G} \xrightarrow{\alpha} G \rightarrow 1$$

we have the standard homology sequence (see for instance Theorem 8.1 and (8.2) on page 202 of [HS]) :

$$(6.11) \quad \begin{aligned} H_2(\tilde{G}, R[G_r]) &\xrightarrow{H_2(\alpha)} H_2(G, R[G_r]) \xrightarrow{\delta_\alpha} K^{\text{ab}} \otimes_{\mathbb{Z}[G]} R[G_r] \rightarrow \\ &\xrightarrow{\iota} H_1(\tilde{G}, R[G_r]) \xrightarrow{H_1(\alpha)} H_1(G, R[G_r]) \rightarrow 0. \end{aligned}$$

Recall that here \tilde{G} , G and K all act on the module $R[G_r]$ via conjugation on G_r . We let $\bar{\xi}_{\tilde{G}}$ (which we recall is denoted $\bar{\nu}_{\tilde{G}}$ in [O5]) denote the composite

$$K_1(R[\tilde{G}]) \xrightarrow{\xi_{\tilde{G}}} H_1(\tilde{G}, R[\tilde{G}_r]) \xrightarrow{\alpha_*} H_1(\tilde{G}, R[G_r]).$$

For $u \in SK_1(R[G])$ we choose a lift $\tilde{u} \in K_1(R[\tilde{G}])$ (see Lemma 6.16) and by Lemma 6.15 we know that $\xi_G(u) = 1$; hence we have

$$(6.12) \quad \bar{\xi}_{\tilde{G}}(\tilde{u}) = \alpha_* \circ \xi_{\tilde{G}}(\tilde{u}) = \xi_G(\alpha(\tilde{u})) = \xi_G(u) = 1.$$

6.e.6. *The map $\bar{\tau}_\alpha$.* We continue with the above notations.

Definition 6.21. *Let $A_\alpha = \ker(R[\tilde{G}] \xrightarrow{\alpha} R[G])$. Then A_α is the $R[\tilde{G}]$ -ideal generated by $(1 - z)$ for all $z \in K$. We now form the \tilde{G} -homology with respect to the exact sequence*

$$0 \rightarrow A_\alpha \rightarrow R[\tilde{G}] \rightarrow R[G] \rightarrow 0$$

to get

$$(6.13) \quad H_1(\tilde{G}, R[G]) \xrightarrow{\partial^\alpha} H_0(\tilde{G}, A_\alpha) \rightarrow H_0(\tilde{G}, R[\tilde{G}]) \rightarrow H_0(\tilde{G}, R[G]) \rightarrow 0$$

and we define

$$\begin{aligned} \bar{H}_0(\tilde{G}, A_\alpha) : &= \frac{H_0(\tilde{G}, A_\alpha)}{\partial^\alpha(H_1(\tilde{G}, R[G]))} = \\ &= \ker(H_0(\alpha) : H_0(\tilde{G}, R[\tilde{G}]) \rightarrow H_0(\tilde{G}, R[G]) = H_0(G, R[G])). \end{aligned}$$

We now define the map $\tau_\alpha : A_\alpha \rightarrow K^{\text{ab}} \otimes_{\mathbb{Z}[G]} R[G_r]$ by the rule that

$$s(1 - z)g \mapsto sz \otimes \alpha(g)_r$$

for $s \in R$, $g \in G$, $z \in K$. We see that τ_α induces a further map, also denoted τ_α , on the covariants

$$\tau_\alpha : H_0(\tilde{G}, A_\alpha) \rightarrow H_0(\tilde{G}, H_1(K, R[G_r])) = K^{\text{ab}} \otimes_{\mathbb{Z}[G]} R[G_r];$$

here we use the fact that $H_1(K, R[G_r]) = K^{\text{ab}} \otimes_{\mathbb{Z}} R[G_r]$ and for $R[G]$ -modules M and N we know that $H_0(\tilde{G}, M \otimes N) = H_0(G, M \otimes N) = M \otimes_{\mathbb{Z}[G]} N$.

Using the above we obtain the exact top row of the following diagram:

$$(6.14) \quad \begin{array}{ccccccc} H_1(\tilde{G}, R[G]) & \xrightarrow{\partial^\alpha} & H_0(\tilde{G}, A_\alpha) & \rightarrow & H_0(\tilde{G}, R[\tilde{G}]) & \rightarrow & H_0(G, R[G]) \\ \downarrow \lambda & & \downarrow \tau_\alpha & & \downarrow \tau_{\tilde{G}} & & \\ H_2(G, R[G_r]) & \xrightarrow{\delta_\alpha} & K^{\text{ab}} \otimes_{\mathbb{Z}[G]} R[G_r] & \xrightarrow{\iota} & H_1(\tilde{G}, R[G_r]) & & \end{array}$$

where λ is defined as follows: note that $H_1(\tilde{G}, R[G])$ is generated by elements $g \otimes sh$ for $s \in R$, and $g \in \tilde{G}$, $h \in G$ with the property that $\alpha(g)$ and h commute; we define $\lambda(g \otimes sh) = \alpha(g) \wedge h \otimes sh_r \in H_2^{\text{ab}}(G, R[G_r])$. The lower row in the diagram is exact by (6.11). Recall that $\overline{H}_0(\tilde{G}, A_\alpha)$ was defined in Definition 6.21 with $\overline{H}_0(\tilde{G}, A_\alpha) = \text{coker}(\partial_\alpha)$ and so τ_α induces a map

$$\overline{\tau}_\alpha : \overline{H}_0(\tilde{G}, A_\alpha) = \ker(H_0(\alpha) : H_0(\tilde{G}, R[\tilde{G}]) \rightarrow H_0(\tilde{G}, R[G])) \rightarrow \frac{K^{\text{ab}} \otimes_{\mathbb{Z}[G]} R[G_r]}{\delta_\alpha(H_2^{\text{ab}}(G, R[G_r]))}.$$

6.e.7. *The diagram.* Noting that by the exact sequence (6.11)

$$\ker(H_1(\tilde{G}, R[G_r]) \xrightarrow{H_1(\alpha)} H_1(G, R[G_r])) = \text{Im}(\iota),$$

we see from the above work that we may now assemble the following diagram:

$$(6.15) \quad \begin{array}{ccc} \ker(H_0(\tilde{G}, R[\tilde{G}]) \xrightarrow{H_0(\alpha)} H_0(G, R[G])) & & \\ \downarrow \overline{\tau}_\alpha & & \\ \frac{K^{\text{ab}} \otimes_{\mathbb{Z}[G]} R[G_r]}{\delta_\alpha(H_2^{\text{ab}}(G, R[G_r]) + (1 - \Psi)H_2(G, R[G_r]))} & \xleftarrow{\delta_\alpha} & \overline{H}_2(G, R[G_r])_\Psi \\ \uparrow (\Psi - 1) \circ \iota^{-1} & & \\ \ker(H_1(\tilde{G}, R[G_r]) \xrightarrow{H_1(\alpha)} H_1(G, R[G_r])) & & \end{array}$$

where by definition (as in the Introduction)

$$\overline{H}_2(G, R[G_r]) \stackrel{\text{def}}{=} \frac{H_2(G, R[G_r])}{H_2^{\text{ab}}(G, R[G_r])}.$$

6.e.8. *The map $\Theta_{R[G]}$.* We continue to assume that R is as in the beginning of §6.c.

Theorem 6.22. *Choose a group extension as in (6.10) with the property that the image of the map $\delta_\alpha : H_2(\tilde{G}, R[G_r]) \rightarrow H_2(G, R[G_r])$ is contained in $H_2^{\text{ab}}(G, R[G_r])$. Note that such extensions exist by Lemma 8.3.iii in [O5]. Let $u \in \text{SK}_1(R[G])$; by Lemma 6.16 we may choose a lift of u denoted $\tilde{u} \in K_1(R[\tilde{G}])$. Mapping u to the value*

$$(6.16) \quad \Theta_{R[G]}(u) = \delta_\alpha^{-1} \left(\overline{\tau}_\alpha \circ v'_{\tilde{G}}(\tilde{u}) + (\Psi - 1) \circ \iota^{-1}(\overline{\xi}_{\tilde{G}}(\tilde{u})) \right) \in \overline{H}_2(G, R[G_r])$$

yields an isomorphism $\Theta_{R[G]} : \text{SK}_1(R[G]) \rightarrow \overline{H}_2(G, R[G_r])_\Psi$ which is independent of the choice of the group extension (6.10).

Remark 6.23. If G is a p -group, then $G_r = \{1\}$ and it is easily seen that the above map $\Theta_{R[G]}$ coincides with the map defined in 3.c.

Proof. First we observe that by the condition on the group extension the map δ_a (in the diagram (6.15)) is a monomorphism: indeed using the exact sequence

$$H_2(\tilde{G}, R[G_r]) \xrightarrow{H_2(\alpha)} H_2(G, R[G_r]) \xrightarrow{\delta_a} K^{\text{ab}} \otimes_{\mathbb{Z}[G]} R[G_r]$$

we see that the induced map (also denoted δ_α)

$$\delta_\alpha : \overline{H}_2(G, R[G_r]) \rightarrow \frac{K^{\text{ab}} \otimes_{\mathbb{Z}[G]} R[G_r]}{\delta_a(H_2^{\text{ab}}(G, R[G_r]))}$$

is injective and hence

$$\delta_\alpha : \overline{H}_2(G, R[G_r])_\Psi \rightarrow \frac{K^{\text{ab}} \otimes_{\mathbb{Z}[G]} R[G_r]}{\delta_a(H_2^{\text{ab}}(G, R[G_r]) + (1 - \Psi)H_2(G, R[G_r]))}$$

is also injective.

Next we want to show that the terms inside the bracket on the right of (6.16) are all defined and that

$$(6.17) \quad \overline{\tau}_\alpha \circ v'_G(\tilde{u}) + (\Psi - 1) \circ \iota^{-1}(\overline{\xi}_{\tilde{G}}(\tilde{u})) \in \ker(i) = \text{Im}(\delta_a).$$

First note that τ_G and $\overline{\xi}_{\tilde{G}}$ admit factorizations

$$\begin{aligned} \tau_{\tilde{G}} &: H_0(\tilde{G}, R[\tilde{G}_r]) \xrightarrow{\omega_{\tilde{G}}} H_1(\tilde{G}, R[\tilde{G}_r]) \xrightarrow{\alpha_*} H_1(\tilde{G}, R[G_r]) = H_1(G, R[G_r]) \\ \overline{\xi}_{\tilde{G}} &: \text{Det}(R[\tilde{G}]^\times) \xrightarrow{\xi_{\tilde{G}}} H_1(\tilde{G}, R[\tilde{G}_r]) \xrightarrow{\alpha_*} H_1(\tilde{G}, R[G_r]) = H_1(G, R[G_r]). \end{aligned}$$

Next observe that as in (6.12)

$$\overline{\xi}_{\tilde{G}}(\tilde{u}) \in \ker(H_1(\alpha) : H_1(\tilde{G}, R[G_r]) \rightarrow H_1(G, R[G_r])) = \text{Im}(\iota).$$

and also note that, as $u \in \text{SK}_1(R[G])$, we have $\text{Det}(u) = 1$, and so $v'_G(u) = 0$ and hence $u \in H_0(G, A_\alpha)$. By the diagram (6.14) we know that $i \circ \tau_\alpha = \tau_G$ and so applying i to the term in the statement of the theorem we get

$$i(\overline{\tau}_\alpha \circ v'_G(\tilde{u}) + (\Psi - 1) \circ \iota^{-1}(\overline{\xi}_{\tilde{G}}(\tilde{u}))) = \overline{\tau}_{\tilde{G}} \circ v_{\tilde{G}}(\tilde{u}) + (\Psi - 1)(\overline{\xi}_{\tilde{G}}(\tilde{u})).$$

We claim that the latter term vanishes: by Proposition 6.20 we know $\omega_{\tilde{G}} \circ v'_G(\tilde{u}) = (1 - \Psi)\overline{\xi}_{\tilde{G}}(\tilde{u})$; then using the factorizations for $\tau_{\tilde{G}}$ and $\overline{\xi}_{\tilde{G}}$ above we see

$$\overline{\tau}_{\tilde{G}} \circ v'_G(\tilde{u}) + (\Psi - 1)(\overline{\xi}_{\tilde{G}}(\tilde{u})) = 0$$

as required. The fact that this value is independent of choices follows exactly as in the argument provided in the proof of Theorem 12.9 of [O5].

To conclude, we know from Theorem 1.6 that $\Theta_{R[G]}$ is an isomorphism if G is a p -group. We now use Theorem 5.5 and Corollary 5.4 to show that $\Theta_{R[G]}$ is an isomorphism if G is \mathbb{Q}_p - p -elementary. The functoriality of $\Theta_{R[G]}$ together with the induction Theorem 4.7 will then show that $\Theta_{R[G]}$ is an isomorphism, in all cases, which agrees with our construction in §6.a.

Suppose now that G is \mathbb{Q}_p - p -elementary and we yet again adopt the notation of Sect. 5. By Theorem 5.5 and Corollary 5.4 we then have isomorphisms

$$\text{SK}_1(R[G]) \cong \oplus_m \text{SK}_1(R[m] \circ P, I_P) \cong \oplus_m H_0(A_m, \text{SK}_1(R[m][H_m]))$$

and

$$\overline{H}_2(G, R[G_r])_\Psi = \overline{H}_2(G, R[C])_\Psi \cong \oplus_m \overline{H}_2(G, R[m])_\Psi \cong \oplus_m H_0(A_m, \overline{H}_2(H_m, R[m]))_\Psi.$$

The latter isomorphism comes from the composition of functors spectral sequence for covariants of G as covariants of H_m followed by covariants of A_m and the fact that $R[m]$ is A_m -free with trivial H_m -action. The result then follows from the fact that by Theorem 3.24, $\Theta_{R[m][H_m]}$ yields a functorial isomorphism from $\mathrm{SK}_1(R[m][H_m])$ to $\overline{H}_2(H_m, R[m])_\Psi$ for each m . \square

7. APPENDIX A: ADAMS OPERATIONS

Throughout this Appendix we assume that R satisfies the conditions imposed in §6.c. The proof of Theorem 6.4 follows the proof of Theorem 1 in [CNT] and the proof of Theorem 1.2 in [T]. There is one crucial difference between these proofs and the proof that we now give for the much more general rings R ; this occurs in the special case when G is a p -group. As in [T] our proof proceeds in five steps. Four steps proceed essentially in the same manner as in [CNT] and [T], and so in these cases we often refer the reader to [T] for details; the fourth step is the case where G is a p -group: this is considerably more involved and it is dealt with in full detail.

For a finite non-ramified extension L of \mathbb{Q}_p we again set $R_L = R \otimes_{\mathbb{Z}_p} \mathcal{O}_L$. Note that if R satisfies the standing hypotheses and the additional above hypothesis, then so does R_L .

For an integer h we define

$$(7.1) \quad M_h(R[G]) = \frac{\psi^h(\mathrm{Det}(R[G]^\times)) \mathrm{Det}(R[G]^\times)}{\mathrm{Det}(R[G]^\times)}.$$

In order to prove Theorem 6.4 it will suffice to show that $M_h(R[G]) = \{1\}$. Our proof proceeds in five steps:

Step 1.

Lemma 7.1. *If G has order prime to p , then $M_h(R[G]) = \{1\}$ for all h .*

Proof. Since G has order prime to p we have isomorphisms

$$\begin{aligned} \mathbb{Z}_p[G] &\cong \prod_i M_{n_i}(\mathcal{O}_i) \\ R[G] &\cong \prod_i M_{n_i}(R \otimes_{\mathbb{Z}_p} \mathcal{O}_i) \end{aligned}$$

for some non-ramified rings of p -adic integers \mathcal{O}_i . Let \mathcal{O}^c denote the valuation ring of the chosen algebraic closure \mathbb{Q}_p^c of \mathbb{Q}_p and set $\Omega_p = \mathrm{Gal}(\mathbb{Q}_p^c/\mathbb{Q}_p)$. Then, as in Proposition 22 on page 23 of [F], we have the isomorphism

$$\mathrm{Det}(R[G]^\times) = \mathrm{Hom}_{\Omega_p}(\mathrm{K}_0(\mathbb{Q}_p^c[G]), (R \otimes_{\mathbb{Z}_p} \mathcal{O}^c)^\times)$$

and the right-hand side is clearly stable under ψ^h for any integer h , since the actions of ψ^h and Ω_p commute. \square

Step 2.

Proposition 7.2. *If G is \mathbb{Q}_p - l -elementary with $l \neq p$, then $M_h(R[G])$ is killed by a power of p .*

Proof. See the proof of Proposition 2.5 on page 105 of [T]. \square

Step 3. Here we suppose that G is a p -group. Recall that we have the natural decomposition

$$\mathrm{Det}(R[G]^\times) = \mathrm{Det}(1 + I(R[G])) \times \mathrm{Det}(R^\times).$$

The factor $\mathrm{Det}(R^\times)$ is clearly stable under Adams operations, and so it will suffice to show that $\mathrm{Det}(1 + I(R[G]))$ is stable under Adams operations. For future use we note the isomorphism

$$(7.2) \quad \begin{aligned} M_h(R[G]) &= \frac{\psi^h(\mathrm{Det}(R[G]^\times))\mathrm{Det}(R[G]^\times)}{\mathrm{Det}(R[G]^\times)} \\ &\cong \frac{\psi^h(\mathrm{Det}(1 + I(R[G])))\mathrm{Det}(1 + I(R[G]))}{\mathrm{Det}(1 + I(R[G]))}. \end{aligned}$$

We shall also use the exact sequence

$$0 \rightarrow p\phi(\mathcal{A}(R[G])) \rightarrow \mathrm{Det}(1 + I(R[G])) \rightarrow 1 + I(R[G^{\mathrm{ab}}]) \rightarrow 1$$

(see (3.8) in [CNT1]). Given $x \in 1 + I(R[G])$, because $g \mapsto g^h$ induces an endomorphism of $R[G^{\mathrm{ab}}]$, because $\mathcal{A}(R[G]) \subset I(R[G])$, and because $I(R[G])$ is contained in the Jacobson radical of $R[G]$, we see that if we set $x = \sum_g x_g g$ and $y = \sum_g x_g g^h \in 1 + I(R[G])$ and if we put

$$\gamma \stackrel{\mathrm{defn}}{=} \psi^h(\mathrm{Det}(x)) \cdot \mathrm{Det}(y)^{-1},$$

then γ is trivial on all abelian characters of G .

Lemma 7.3. *We have $\nu_G(\gamma) \in p\phi(\mathcal{A}(R[G])) = \nu_G(\mathrm{Det}(1 + \mathcal{A}(R[G])))$.*

Proof. Since for an abelian character χ of G we have

$$\chi(\nu_G(\gamma)) = \log(\gamma(p\chi - \psi^p\chi)) = 0,$$

it follows that $\nu_G(\gamma) \in p\phi(\mathcal{A}(R[G])) \otimes \mathbb{Q}_p$. Let $\nu_G(\mathrm{Det}(x)) = \sum_{c \in C_G} \lambda_c c$ with $\lambda_c \in pR$ by Theorem 3.3. We now show that

$$\nu_G(\psi^h \mathrm{Det}(x)) = \sum_{c \in C_G} \lambda_c c^h.$$

This follows from the fact that, for each character χ of G , we have the two equalities

$$\begin{aligned} \chi(\nu_G(\psi^h \mathrm{Det}(x))) &= \log((\psi^h \mathrm{Det}(x))(p\chi - \psi^p\chi)) \\ &= \log(\mathrm{Det}(x)(p\psi^h\chi - \psi^p\psi^h\chi)) = \log(\mathrm{Det}(x)(p\psi^h\chi - \psi^{ph}\chi)) \end{aligned}$$

and

$$\begin{aligned} \chi(\sum_{c \in C_G} \lambda_c c^h) &= \psi^h\chi(\sum_{c \in C_G} \lambda_c c) \\ &= \psi^h\chi(\nu_G(\mathrm{Det}(x))) = \log(\mathrm{Det}(x)(p\psi^h\chi - \psi^{ph}\chi)). \end{aligned}$$

This shows that $\nu_G(\gamma) \in pR[C_G]$ and we have seen that $p\phi(\mathcal{A}(R[G])) \otimes \mathbb{Q}_p$ and so $\nu_G(\gamma) \in p\phi(\mathcal{A}(R[G]))$. To conclude we note that by Theorem 3.4 we know that $p\phi(\mathcal{A}(R[G])) = \nu_G(\text{Det}(1 + \mathcal{A}(R[G])))$. \square

Lemma 7.4. *The group*

$$\ker(\nu_G : \psi^h \text{Det}(1 + I(R[G])) \cdot \text{Det}(1 + I(R[G])) \rightarrow N[C_G])$$

is p -power torsion.

Proof. In (3.3) we have seen that

$$\ker(\nu_G : \text{Det}(1 + I(R[G])) \rightarrow pR[C_G]) = \text{Det}(G)$$

which is p -power torsion. The result then follows from part (a) of Proposition 7.8 in Step 5 (which only uses Step 2) that $M_h(R[G])$ is p -power torsion. \square

With the above notation we again consider $\gamma = \psi^h(\text{Det}(x)) \cdot \text{Det}(y)^{-1}$. By Lemma 7.3 we know that $\nu_G(\gamma) \in p\phi(\mathcal{A}) = \text{Det}(1 + \mathcal{A})$ and so by Lemma 7.4 we know that we can write

$$\gamma = \text{Det}(z) \cdot t$$

with $z \in 1 + \mathcal{A}$, and

$$t \in \ker(\nu_G : \psi^h(\text{Det}(1 + I(R[G])) \cdot \text{Det}(1 + I(R[G])) \rightarrow N[C_G])$$

so that by Lemma 7.4 t is p -power torsion and is trivial on abelian characters of G . Therefore we may write

$$\psi^h(\text{Det}(x)) = \text{Det}(zy) \cdot t.$$

Thus, to prove that $M_h(R[G]) = \{1\}$, it will suffice to show that $t = 1$.

First note that if χ is a character of G , whose degree is denoted $\chi(1)$, then, as G is a p -group, $\chi - \chi(1) \in \ker d_p$ and so as in the methods used for Step 2 (see page 106 in [T]), we know that

$$\psi^h \text{Det}(x)(\chi - \chi(1)) = \text{Det}(x)(\psi^h \chi - \psi^h \chi(1)) \equiv 1 \pmod{P^c}$$

where P^c denotes the $R \otimes_{\mathbb{Z}_p} \mathcal{O}^c$ -ideal generated by the maximal ideal of \mathcal{O}^c ; therefore, since $\text{Det}(zy)(\chi - \chi(1)) \equiv 1 \pmod{P^c}$, we deduce that

$$t(\chi) = t(\chi - \chi(1)) \equiv 1 \pmod{P^c}.$$

Let T_R denote the subgroup of p -torsion elements $\text{Hom}_{\Omega_p}(\mathbb{K}_0(\mathbb{Q}_p^c[G]), (R \otimes \mathcal{O}^c)^\times)$ which are trivial on the abelian characters of G . We can now use the above work to define a homomorphism $\xi : M_h(R[G]) \rightarrow T_R$ by the rule that

$$\xi(\psi^h \text{Det}(x) \text{Det}(R[G])^\times) = t.$$

Note that by the exact sequence (3.3) we know that

$$\text{Det}(1 + I(R[G])) \cap T_R = \text{Det}(G) \cap T_R = \{1\}.$$

This map is well-defined and injective. For instance to see that ξ is injective, with the obvious notation suppose that

$$\xi(\psi^h \text{Det}(x)) = t = \xi(\psi^h \text{Det}(x'))$$

then

$$\psi^h \text{Det}(x) \text{Det}(y)^{-1} = t = \psi^h \text{Det}(x') \text{Det}(y')^{-1}$$

and so $\psi^h \text{Det}(xx'^{-1}) = \text{Det}(yy'^{-1}) \in \text{Det}(R[G]^\times)$.

Lemma 7.5. *Let L denote a finite non-ramified extension of \mathbb{Q}_p in \mathbb{Q}_p^c and put $R_L = R \otimes_{\mathbb{Z}_p} \mathcal{O}_L$. From Lemma 6.1 in [CNT1] we know that there is a decomposition of R -algebras*

$$R_L = \prod_{i=1}^{n(L)} R_{L,i}$$

where the $R_{L,i}$ are integral domains which satisfy the Standing Hypotheses. Then the numbers $n(L)$ are bounded as L ranges over all finite non-ramified extensions of \mathbb{Q}_p in \mathbb{Q}_p^c .

Proof. Let $\mathcal{O}_p^{\text{nr}}$ denote the valuation ring of the maximal non-ramified extension of \mathbb{Q}_p in \mathbb{Q}_p^c . By lifting idempotents (see for instance Theorem 6.7 on page 123 of [CR1]) we know that there is natural bijection between the idempotents of $\overline{R} \otimes_{\mathbb{F}_p^c}$ and $R \otimes_{\mathbb{Z}_p} \mathcal{O}_p^{\text{nr}}$; therefore, by the additional hypotheses introduced in 6.c, $R \otimes_{\mathbb{Z}_p} \mathcal{O}_p^{\text{nr}}$ contains only a finite number of orthogonal idempotents. The result then follows. \square

Lemma 7.6. *Let ζ_{p^n} denote a primitive p^n -th root of unity in \mathbb{Q}_p^c for $n > 0$. Suppose $E \subset \mathbb{Q}_p(\zeta_{p^n})$. Then*

- (a) $R \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ is an integral domain;
- (b) $\mu_{p^\infty}(R \otimes_{\mathbb{Z}_p} \mathcal{O}_E) = \mu_{p^\infty}(\mathcal{O}_E)$.

Proof. (a) Recall that we write N for the field of fractions of the normal domain R . Since R and \mathcal{O}_E are flat over \mathbb{Z}_p we know that $R \otimes_{\mathbb{Z}_p} \mathcal{O}_E \subset N \otimes_{\mathbb{Q}_p} E$ and so it will suffice to show that N and E are linearly disjoint over \mathbb{Q}_p . Let π_E denote a uniformising parameter for E and suppose for contradiction that

$$[NE : N] = m < [E : \mathbb{Q}_p].$$

Then $\text{Norm}_{NE/N}(\pi_E)$ belongs to R and has p -valuation $[NE : N][E : \mathbb{Q}_p]^{-1} < 1$ which contradicts the fact that p is prime in R .

To prove (b) suppose for contradiction that $\zeta_{p^r} \in \mu_{p^\infty}(R \otimes_{\mathbb{Z}_p} \mathcal{O}_E)$ but $\zeta_{p^r} \notin E$. We put $M = E(\zeta_{p^r})$ which strictly contains E and is totally ramified over \mathbb{Q}_p and we let π_M denote a uniformising parameter for M . Then using (a) we see that $\text{Norm}_{ME/N}(\pi_M)$ has p -valuation

$$[NE : N][M : \mathbb{Q}_p]^{-1} = [E : \mathbb{Q}_p][M : \mathbb{Q}_p]^{-1} < 1$$

which again contradicts the fact that p is prime in R . \square

We now conclude the proof of Step 3. Suppose that the torsion group T_R has exponent p^e , let L'/\mathbb{Q}_p denote a non-ramified extension with $n(L')$ maximal as in

Lemma 7.5. We write $\mathcal{N}_{L'/\mathbb{Q}_p}$ for the co-restriction (or norm map) of L'/\mathbb{Q}_p . Then we can find a non-ramified extension L of L' of degree p^e . By Lemma 7.6 we know that $T_{R_L} = T_R$ and so

$$\mathcal{N}_{L/\mathbb{Q}_p}(T_{R_L}) = \mathcal{N}_{L'/\mathbb{Q}_p} \circ \mathcal{N}_{L/L'}(T_{R_L}) = \mathcal{N}_{L'/\mathbb{Q}_p}(T_{R_L}^{p^e}) = 1.$$

We then have a diagram

$$\begin{array}{ccccc} 1 & \rightarrow & M_h(R_L[G]) & \xrightarrow{\xi} & T_{R_L} \\ & & \downarrow \mathcal{N}_{L/\mathbb{Q}_p} & & \downarrow 0 \\ 1 & \rightarrow & M_h(R[G]) & \xrightarrow{\xi} & T_R. \end{array}$$

But by Theorem 6.5 we know that $\mathcal{N}_{L/\mathbb{Q}_p}(1 + I(R_L[G])) = \mathcal{N}_{L/\mathbb{Q}_p}(1 + I(R[G]))$; hence $\mathcal{N}_{L/\mathbb{Q}_p}(M_h(R_L[G])) = M_h(R[G])$, and we have therefore shown that for a p -group G we have $M_h(R[G]) = \{1\}$. \square

Step 4.

Proposition 7.7. *For each \mathbb{Q}_p - p -elementary group G , and for all integers h ,*

$$M_h(R[G]) = \{1\}.$$

Proof. This is essentially the same as the proof of the corresponding statement given in the first part of Sect. 3 in Ch. 9 of [T]. \square

Step 5.

Proposition 7.8. (a) *Given an integer h and a prime number $l \neq p$, then, given a finite group G , we can find an integer m_l , which is not divisible by l , so that $m_l M_h(R[G]) = \{1\}$ for all h , and so $M_h(R[G])$ is killed by a power of p .*

(b) *Given an integer h , then $M_h(R[G])$ is killed by an integer which is coprime to p .*

(c) *$M_h(R[G]) = \{1\}$ for an arbitrary finite group G .*

Proof. The arguments are similar to the proof of the corresponding statement in [T]. For example for (a) we can start the argument by observing that by Theorem 28 in [S] we can write

$$m_l = \sum_{H \in \mathcal{E}_l(\mathbb{Q}_p)} n_H \text{Ind}_H^G \theta_H$$

and then use Step 2.

Part (b) follows similarly using Step 4, and then (c) follows immediately from (a) and (b). \square

As above, we now see that this implies Theorem 6.4.

8. APPENDIX B.

8.a. Proof of Corollary 1.8. The notation here is as in the Introduction. For the convenience of the reader we recall some of the set-up. We let $\{C_i\}_{i \in I}$ denote the set of G -conjugacy classes in G_r , let g_i be a chosen group element whose conjugacy class lies in C_i and let G_i denote the centralizer of g_i in G ; then we have a disjoint union

decomposition $G_r = \sqcup_i g_i^{G/G_i}$. Next consider the action of Ψ on the $\{C_i\}_{i \in I}$. We may view this action as an action on I and we let J denote the set of orbits of the action of Ψ on I ; for $j \in J$ we let n_j denote the cardinality of j . We then obtain a further disjoint union decomposition

$$G_r = \sqcup_j \sqcup_{m=1}^{n_j} (g_{i_j}^{G/G_{i_j}})^{\Psi^m}$$

where i_j denotes a chosen element of the orbit j . This induces the decomposition of homology groups

$$\begin{aligned} H_2(G, R[G_r]) &= H_2(G, \mathbb{Z}[G_r]) \otimes R \\ &= \oplus_i H_2(G_i, \mathbb{Z}g_i) \otimes R \end{aligned}$$

and hence a decomposition

$$\begin{aligned} H_2(G, R[G_r])_{\Psi} &= (\oplus_j \oplus_{m=1}^{n_j} H_2(G_{i_j}, \mathbb{Z}g_{i_j}^{\Psi^m}) \otimes R)_{\Psi} \\ &= (\oplus_j (H_2(G_{i_j}, \mathbb{Z}) \otimes \frac{\mathbb{Z}[\Psi]}{(\Psi^{n_j} - 1)}) \otimes R)_{\Psi} \\ &= \oplus_j [H_2(G_{i_j}, \mathbb{Z}) \otimes R_{\Psi^{n_j}}]_{\Psi} \\ &= \oplus_j \left[H_2(G_{i_j}, \mathbb{Z}) \otimes \frac{R}{(F-1)R} \right]. \end{aligned}$$

We now show that we have a similar decomposition

$$(8.3) \quad (H_2^{\text{ab}}(G, R[G_r]))_{\Psi} = \bigoplus_j \left[H_2^{\text{ab}}(G_{i_j}, \mathbb{Z}) \otimes \frac{R}{(F-1)R} \right]$$

and hence the decomposition

$$(\overline{H}_2(G, R[G_r]))_{\Psi} = \bigoplus_j \left[\overline{H}_2(G_{i_j}, \mathbb{Z}) \otimes \frac{R}{(F-1)R} \right].$$

To this end we observe that for an abelian subgroup A of G , if as previously, we write $\{A_r\} = \{a_1, \dots, a_k\}$, then, because A acts trivially on A_r , we can write

$$H_2(A, R[A_r]) = \oplus_{l=1}^k H_2(A, Ra_l).$$

If a_l is a conjugate of $g_{i_j(l)}^{p^m}$, with say $a_l^h = g_{i_j(l)}^{p^m}$ for $h \in G$; then, since A^h centralizes $g_{i_j(l)}^{p^m}$, we know that A^h centralizes $g_{i_j(l)}$, and so we have $A^h \subset G_{i_j(l)}$ and hence

$$\begin{aligned} \text{Cor}_A^G(H_2(A, Ra_l)) &= \text{Cor}_{A^h}^G(H_2(A^h, Ra_l^h)) \subset \\ &\subset H_2(G, \sum_{\gamma \in G_{i_j(l)} \setminus G} \sum_{m=1}^{n_j} Ra_l^{h\gamma\Psi^m}) = H_2(G_{i_j}, \sum_{m=1}^{n_j} Rg_{i_j(l)}^{\Psi^m}). \end{aligned}$$

Conversely we know that $H_2^{\text{ab}}(G_{i_j}, R)$ is generated by the images under corestriction of the $H_2(B, R)$ for maximal abelian subgroups B of G_{i_j} . Since g_{i_j} is centralized by such a maximal abelian subgroup of G_{i_j} , we see that B must contain g_{i_j} ; therefore we have the inclusion

$$\text{Cor}_B^{G_{i_j}}(H_2(B, R)) = \text{Cor}_B^{G_{i_j}}(H_2(B, Rg_{i_j})) \subset H_2^{\text{ab}}(G_{i_j(l)}, \sum_{m=1}^{n_j} Rg_{i_j(l)}^{\Psi^m})$$

which establishes (8.3), as required. \square

8.b. **Proof of Corollaries 1.10, 1.11.** Recall that $F(t) = t^p$. We start with Cor. 1.10: Note that $W\{\{t\}\} = W\langle\langle t^{-1} \rangle\rangle + tW[[t]]$. Since $(1 - F)tW[[t]] = tW[[t]]$, we see that

$$\frac{W\langle\langle t^{-1} \rangle\rangle}{(1 - F)W\langle\langle t^{-1} \rangle\rangle} \cong \frac{W\{\{t\}\}}{(1 - F)W\{\{t\}\}}$$

and the result follows from Corollary 1.9. \square

Now we consider the proof of Cor. 1.11:

We have

$$W\{\{t\}\} = t^{-1}W\langle\langle t^{-1} \rangle\rangle \oplus W[[t]]$$

and we get

$$\frac{W\{\{t\}\}}{(1 - F)W\{\{t\}\}} = \frac{t^{-1}W\langle\langle t^{-1} \rangle\rangle}{(1 - F)t^{-1}W\langle\langle t^{-1} \rangle\rangle} \oplus \frac{W}{(1 - F)W}.$$

This certainly shows that the map

$$\frac{W[[t]]}{(1 - F)W[[t]]} \rightarrow \frac{W\{\{t\}\}}{(1 - F)W\{\{t\}\}}$$

is injective.

We write W_m for $W/p^m W$. We have

$$W_m((t)) = t^{-1}W_m[t^{-1}] \oplus W_m[[t]],$$

and we note that

$$t^{-1}W_m[t^{-1}] = \oplus_{p \nmid k > 0} W_m t^{-k} \oplus_{k > 0} (1 - F)W_m t^{-k} = \oplus_{p \nmid k > 0} W_m t^{-k} \oplus (1 - F)t^{-1}W_m[t^{-1}];$$

therefore, for each $m > 0$, we have shown

$$\frac{W_m((t))/(1 - F)W_m((t))}{W_m[[t]]/(1 - F)W_m[[t]]} = \oplus_{p \nmid k > 0} W_m t^{-k}.$$

Using the Mittag-Leffler condition twice we get

$$\begin{aligned} \frac{W\{\{t\}\}/(1 - F)W\{\{t\}\}}{W[[t]]/(1 - F^n)W[[t]]} &= \frac{\varprojlim_m W_m((t))/(1 - F)W_m((t))}{\varprojlim_m W_m[[t]]/(1 - F)W_m[[t]]} \\ &= \varprojlim_m \frac{W_m((t))/(1 - F)W_m((t))}{W_m[[t]]/(1 - F)W_m[[t]]} \\ &= \varprojlim_m \oplus_{p \nmid k > 0} W_m t^{-k} \end{aligned}$$

which is torsion free. \square

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